

# Gravitational waves from stochastic relativistic sources: primordial turbulence and magnetic fields

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The power spectrum of a homogeneous and isotropic stochastic variable, characterised by a finite correlation length, does in general not vanish on scales larger than the correlation scale. If the variable is a divergence free vector field, we demonstrate that its power spectrum is blue on large scales. Accounting for this fact, we compute the gravitational waves induced by an incompressible turbulent fluid and by a causal magnetic field present in the early universe. The gravitational wave power spectra show common features: they are both blue on large scales, and peak at the correlation scale. However, the magnetic field can be treated as a coherent source and it is active for a long time. This results in a very effective conversion of magnetic energy in gravitational wave energy at horizon crossing. Turbulence instead acts as a source for gravitational waves over a time interval much shorter than a Hubble time, and the conversion into gravitational wave energy is much less effective. We also derive a strong constraint on the amplitude of a primordial magnetic field when the correlation length is much smaller than the horizon.

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## I. INTRODUCTION

Gravitational waves, once emitted, propagate freely in spacetime (affected only by the expansion of the universe); they do not interact and are not absorbed. They hence provide a direct probe of physical processes that took place at the time of their generation, in the very early universe. The potential detection of a gravitational wave background of primordial origin provides an observational signal of sources that have disappeared by now. Primordial sources of gravitational waves are for example inflation, cosmic topological defects, a pre big bang phase of expansion, primordial phase transitions, a phase of turbulence in the cosmic fluid and primordial magnetic fields. Gravitational waves produced during inflation [1], by topological defects [2] or during a pre-big bang phase [3] have been studied extensively. In this paper we will consider the last two sources which can also be associated with a phase transition.

Gravitational waves are most probably the only remnant from a period of turbulence in the primordial plasma. In order to be maintained, turbulence necessitates a continuous injection of energy in the absence of which it dissipates rapidly leaving no other trace behind. A possible source for this energy input could come from the false vacuum decay during a first order phase transition.

Primordial magnetic fields could be at the origin of the ubiquitous magnetic fields observed in galaxies and clusters today. Their imprint in the form of gravitational waves can confirm their presence in the primordial uni-

verse also on scales on which magnetic fields are completely dissipated today.

In the following we study in detail the generation of gravitational waves by these two sources and we compare the results obtained. The sources share common properties: they are both related to divergence-free vector fields, and they both obey a relativistic equation of state so that they evolve like radiation with the expansion of the universe. They are also both correlated over a finite correlation length, if one accounts only for causally created magnetic fields and turbulence. However, turbulence lasts for a short time (*e.g.* the duration of the phase transition) compared to the evolution of the universe and it acts incoherently as a source of gravitational waves, while the magnetic field is a coherent source active in principle up to the time of matter and radiation equality.

In the next section we demonstrate that the power spectra of both sources are blue on scales larger than the correlation scale. This implies that turbulence in the primordial universe has a Batchelor spectrum on large scales. Moreover, it gives rise to a background of gravitational waves also on these large scales, which has been disregarded so far. In section III we model the power spectrum of turbulence and of the turbulent anisotropic stress, and we analyse the creation of gravitational waves. We propose a new method to perform this analysis, avoiding an error in the gravitational wave dispersion relation which is found in previous works. We find that the gravitational wave spectrum peaks at a frequency corresponding to the *size* of the largest turbulent eddies. In Section IV we compute the gravitational waves induced by a causal magnetic field correlated on a scale which may be much smaller than the horizon, and we constrain the magnetic fields using the nucleosynthesis bound. Finally, in Section V, we compare the two results,

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concluding that a radiation-like, long acting, coherent source like the magnetic field is more efficient in generating gravitational waves on large scales than a rapidly dissipating, incoherent source such as turbulence. We believe that this result is valid beyond the two examples discussed here.

We consider a Friedmann universe with flat spatial sections,

$$ds^2 = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j]. \quad (1)$$

The variables  $\mathbf{x}$ ,  $r$ ,  $L$  etc. denote comoving distances and  $\mathbf{k}$  is the comoving wave vector. The density parameter is always scaled to today,  $\Omega_X(\eta) \equiv \rho_X(\eta)/\rho_c(\eta_0)$ , where the index  $_0$  indicates the present time. For relativistic species we have therefore  $\Omega_X(\eta) = \Omega_X(\eta_0)/a^4(\eta)$ ; we normalise  $a(\eta_0) = 1$  and sometimes denote the present value of a density parameter simply by  $\Omega_X(\eta_0) \equiv \Omega_X$ ; likewise,  $\rho_c = \rho_c(\eta_0)$ .  $\mathcal{H} = \frac{\dot{a}}{a}$  denotes the conformal Hubble parameter. The radiation energy density today is taken to be  $\Omega_{\text{rad}}(\eta_0) \equiv \Omega_{\text{rad}} = 4.2 \times 10^{-5}$  [4]. We use the expression

$$a(\eta) = H_0 \eta \left( \frac{H_0 \eta}{4} + \sqrt{\Omega_{\text{rad}}} \right)$$

for the scale factor which is strictly true only in a flat matter–radiation universe. But since we are mainly concerned about the physics during the radiation dominated phase of the universe, neglecting the cosmological constant  $\Lambda$  is irrelevant for our purpose. Values of conformal time  $\eta \ll 2(\sqrt{2} - 1)\sqrt{\Omega_{\text{rad}}}/H_0 = \eta_{\text{eq}}$  denote the radiation dominated phase, while during the matter dominated phase  $\eta \gg \eta_{\text{eq}}$ .

## II. THE POWER SPECTRUM OF DIVERGENCE FREE VECTOR FIELDS AT LARGE SCALES

We want to discuss gravitational waves generated from the energy momentum tensor of divergence-free vector fields. In this section we derive the generic form of the spectrum for such vector fields.

We consider a stochastic vector field  $\mathbf{v}(\mathbf{x})$ , statistically homogeneous and isotropic, hence  $\langle v_i(\mathbf{x}) \rangle = 0$ . The general form of its two point correlation tensor is (see for example [5])<sup>1</sup>

$$\begin{aligned} b_{ij}(\mathbf{x}) &\equiv \langle v_i(\mathbf{y}) v_j(\mathbf{y} + \mathbf{x}) \rangle = \\ &= \Sigma(r)(\delta_{ij} - \hat{x}_i \hat{x}_j) + \Gamma(r) \hat{x}_i \hat{x}_j, \end{aligned} \quad (2)$$

where  $\hat{\mathbf{x}}$  is the unit vector in direction  $\mathbf{x}$ , and  $r$  is the length of  $\mathbf{x}$ . Here  $\Sigma(r)$  and  $\Gamma(r)$  denote the correlation

functions in directions perpendicular and parallel to  $\hat{\mathbf{x}}$ . If the stochastic vector field is divergence-free (for example in the case of the turbulent velocity field of an incompressible fluid, or in the case of a magnetic field), the correlation functions  $\Sigma(r)$  and  $\Gamma(r)$  are related. It is easy to verify that  $\frac{\partial v_i(\mathbf{x})}{\partial x_i} = 0$  yields

$$\frac{d\Gamma(r)}{dr} = \frac{2}{r} (\Sigma(r) - \Gamma(r)). \quad (3)$$

Homogeneity implies that the Fourier transform of  $\mathbf{v}$  for different wave vectors  $\mathbf{k} \neq \mathbf{q}$  is uncorrelated, so that the power spectrum is of the form

$$\langle v_i(\mathbf{k}) v_j^*(\mathbf{q}) \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{q}) P_{ij}(\mathbf{k}), \quad (4)$$

$$\text{where } P_{ij}(\mathbf{k}) = \int d^3x e^{i\mathbf{k} \cdot \mathbf{x}} b_{ij}(\mathbf{x}). \quad (5)$$

Isotropy and  $\nabla \cdot \mathbf{v} = 0$  require the following simple form for the power spectrum  $P_{ij}(\mathbf{k})$

$$P_{ij}(\mathbf{k}) = P(k)(\delta_{ij} - \hat{k}_i \hat{k}_j). \quad (6)$$

The trace of Eq. (5) together with Eq. (2) lead to

$$P(k) = \int d^3x e^{i\mathbf{k} \cdot \mathbf{x}} \left( \Sigma(r) + \frac{1}{2} \Gamma(r) \right). \quad (7)$$

In the expanding (non inflationary) universe, any causal random process is characterised by a finite correlation length  $L$ , which is at most the size of the horizon at the epoch at which the process is occurring,  $L \leq \eta_*$ . The correlation function  $b_{ij}(\mathbf{x})$  is generally expected to decay at large distances, but in the case of the expanding universe, causality even requires that it vanishes identically on scales larger than the horizon  $\eta_*$  (if  $v_i(\mathbf{x})$  is a classical field, see [7]). Therefore, in the case of interest to us, the correlation function  $b_{ij}(\mathbf{x})$  of a causal process is always a function with compact support. This implies that its Fourier transform must be an analytic function in the variable  $\mathbf{k}$ . Hence, the power spectrum (6) must be analytic in  $\mathbf{k}$ . In order for  $P(k)(\delta_{ij} - \hat{k}_i \hat{k}_j)$  to be an analytic function of the components  $k_i$  also at  $\mathbf{k} = 0$ , we have to impose  $P(k \rightarrow 0) \propto k^n$ , with  $n$  an even positive integer,  $n \geq 2$ . From Eqs. (5) and (6) it follows that  $P(k)$  is given by

$$P(k) = \frac{1}{2} \int d^3x e^{i\mathbf{k} \cdot \mathbf{x}} b_{ii}(r) = 2\pi \int_0^\infty dr r^2 \frac{\sin(kr)}{kr} b_{ii}(r). \quad (8)$$

For small  $k$  we obtain up to order  $k^2$

$$\begin{aligned} P(k \rightarrow 0) &= 2\pi \int_0^\infty dr r^2 b_{ii}(r) \\ &\quad - \frac{\pi}{3} k^2 \int_0^\infty dr r^4 b_{ii}(r) + \mathcal{O}(k^4). \end{aligned} \quad (9)$$

Analyticity of  $P_{ij}(\mathbf{k})$  requires the first term on the right hand side of Eq. (9) to vanish. This result can also be

<sup>1</sup> This form of the correlator is not completely general in the sense that it does not contain helicity, i.e. a component which is odd under parity (see [6]). We neglect parity violating terms in this work.

obtained directly from Eq. (3) using the fact that  $\Gamma(r)$  vanishes for sufficiently large distances  $r$ . Indeed, using (3) one finds

$$\int_0^\infty dr r^2 b_{ii}(r) = r^3 \Gamma(r)|_0^\infty = 0, \quad (10)$$

since  $\Gamma(r)$  has compact support. One is thus left with

$$P(k \rightarrow 0) = \left( \frac{2\pi}{3} \int_0^\infty dr r^4 \Gamma(r) \right) k^2 + \mathcal{O}(k^4). \quad (11)$$

These considerations apply both to the power spectrum of a magnetic field, if it has been generated in the early universe by a causal process [8], and to the power spectrum of turbulent motions in the primordial fluid, which can be generated for example at the end of a first order phase transition (if one assumes that the fluid is incompressible) [9, 10, 11, 12].

### III. GRAVITATIONAL WAVES FROM TURBULENCE

A period of turbulence in the primordial universe can leave an observable trace today in the form of a stochastic background of gravitational waves. The most natural source of turbulence in the early universe is a first order phase transition (see [13] for another potential source of turbulence), where bubbles of the broken phase nucleate and expand into the symmetric phase. Therefore we concentrate on this case. We parameterise the specifics of the phase transition following [9, 10, 11, 12]. The temperature and the value of the scale factor at which it takes place are denoted by  $T_*$ ,  $a_*$ ; the comoving size of the largest bubbles at the moment when they collide is  $L$ . This length corresponds to the largest scale on which turbulence develops (the size of the largest eddies). The ratio of the false vacuum energy density to the thermal energy density of the universe at the moment of the transition is  $\alpha = \rho_{\text{vac}}/\rho_*$ ; and the velocity of the bubble walls  $v_b(\alpha)$ . Finally, the fraction of vacuum energy density which is converted into kinetic energy, turbulence, of the fluid is denoted by  $\kappa = \rho_{\text{kin}}/\rho_{\text{vac}}$ .

To generate a considerable amount of gravitational waves, one needs to move big masses rapidly. This can be achieved when a first order phase transition proceeds through detonation of the broken phase bubbles. In this case the bubble walls propagate faster than the speed of sound  $1/\sqrt{3} < v_b(\alpha) < 1$ , and there is a large concentration of energy at the bubble walls. The phase transition is completed in a (conformal) time interval of the order of  $\Delta\eta_* \simeq L/v_b(\alpha)$  which is much smaller than one Hubble time  $\Delta\eta_* \mathcal{H}_* \ll 1$ . Typical values for  $\Delta\eta_* \mathcal{H}_*$  for the electroweak phase transition are 0.001 – 0.01 (see [9, 10, 11, 12] and references therein).

A first order phase transition causes injection of kinetic energy in the primordial fluid at a characteristic scale  $L$  corresponding to the size of the largest bubbles

at the moment when they collide. Therefore, the primordial plasma is stirred on the scale  $L \ll \mathcal{H}_*^{-1}$ . Since the Reynolds number is very high in the early universe, turbulent motions are generated [14]. We assume that the generated turbulence is homogeneous, isotropic, fully developed and stationary. Then, since  $L$  is the principal turbulence scale, on smaller scales (larger wave numbers)  $k > L^{-1}$  an energy cascade is established which proceeds from large to small scales. Eventually one expects the formation of a Kolmogorov spectrum in the ‘inertial range’  $L^{-1} \ll k \ll \lambda^{-1}$ , where  $\lambda$  denotes the dissipation scale [5]. In the next section we derive an approximate form of the turbulent power spectrum which is valid at all wavenumbers (including large scales, small wavenumbers  $k < L^{-1}$ ).

#### A. Turbulence power spectrum

The variable  $v_i(\mathbf{x})$  represents the fluid velocity at the position  $\mathbf{x}$ , and we denote its power spectrum by  $P_v(k)$ . Following [10] we define the total kinetic energy per unit enthalpy of the fluid as

$$\frac{\rho_{\text{kin}}}{\rho_* + p_*} \equiv \frac{1}{2} \langle v_i(\mathbf{x}) v_i(\mathbf{x}) \rangle = \frac{1}{2} b_{ii}(0), \quad (12)$$

where  $\rho_*$  and  $p_*$  denote the energy density and pressure at the time of the turbulence. This non-relativistic approximation is strictly valid only for  $\langle v^2 \rangle \ll 1$ : the Kolmogorov theory of turbulence has been formulated and verified only in this regime. However, we shall use it also in the relativistic regime, imposing the upper bound  $\langle v^2 \rangle \leq 1/3$  ([15]) where it still gives the correct order of magnitude. The details of the Kolmogorov spectral shape are probably not valid in the relativistic regime, but we follow here earlier literature on the subject, see [9, 10, 11, 12]. As usual, we also define the energy dissipated per unit time and unit enthalpy by [5, 16, 17]

$$\varepsilon = -\frac{d}{d\eta} \frac{\langle v^2 \rangle}{2}. \quad (13)$$

The turbulent velocity power spectrum can be evaluated from Eq. (8), if the shape of the two point correlation function  $b_{ii}(r)$  is known. If the turbulence is homogeneous and isotropic, in the inertial range  $\lambda \ll r \ll L$  the correlation function is given by the Kolmogorov two thirds law  $(\varepsilon r)^{2/3}$ . In the dissipation range  $r \ll \lambda$ , the fluid motions are regular and one can Taylor expand the velocity field as a function of  $r$ . This entails that the correlation function is proportional to  $r^2$  [5]. Moreover, the symmetric tensor field  $b_{ij}(\mathbf{x})$  has compact support and its divergence vanishes. On this basis, we make the following Ansatz for the transverse correlation function

$$\Sigma(r) \simeq \begin{cases} \frac{\langle v^2 \rangle}{3} + \Sigma_0 \left( \frac{r}{\lambda} \right)^2 & \text{for } 0 < r \leq \lambda \\ \frac{\langle v^2 \rangle}{3} + \Sigma_0 \left( \frac{r}{\lambda} \right)^{2/3} & \text{for } \lambda \leq r < L \\ 0 & \text{otherwise,} \end{cases} \quad (14)$$

with  $\Sigma_0$  an as yet unspecified constant. From the divergence-free condition (3) we now determine the radial correlation function

$$\Gamma(r) \simeq \begin{cases} \frac{\langle v^2 \rangle}{3} + \frac{\Sigma_0}{2} \left(\frac{r}{\lambda}\right)^2 & \text{for } 0 < r \leq \lambda \\ \frac{\langle v^2 \rangle}{3} - \frac{\Sigma_0}{4} \left(\frac{\lambda}{r}\right)^2 + \frac{3}{4} \Sigma_0 \left(\frac{r}{\lambda}\right)^{2/3} & \text{for } \lambda \leq r \leq L \\ 0 & \text{otherwise.} \end{cases}$$

Eq. (3) also requires that  $\Gamma$  be continuous, so that  $\Sigma$  does not contain a singularity ( $\Gamma$  may, however, have a kink which leads to a discontinuity in  $\Sigma$ ). The continuity of  $\Gamma(r)$  fixes  $\Sigma_0$  in terms of  $\lambda$ ,  $\langle v^2 \rangle$  and  $L$ :  $\Gamma(r \geq L) = 0$  implies

$$\Sigma_0 \simeq -\frac{4}{9} \left(\frac{\lambda}{L}\right)^{2/3} \langle v^2 \rangle \quad (15)$$

(where we have neglected a term of order  $(\lambda/L)^2$ ). The radial correlation function is non negative, continuous and identically zero on scales larger than the correlation scale. The transverse correlation function starts positive, passes through zero inside the inertial range at  $r = (3/4)^{3/2}L$ , and has a jump of  $-(1/9)\langle v^2 \rangle$  at  $r = L$  in order to satisfy the condition of compact support. This jump corresponds to a kink in  $\Gamma(r)$  for  $r = L$ . The integral of  $b_{ii}(r)$  over space vanishes, so that the power spectrum satisfies  $P_v(k=0) = 0$ . Fourier transforming the correlation function given above, one finds

$$P_v(k) \simeq 2\pi \langle v^2 \rangle \times \begin{cases} \frac{2}{765} L^5 k^2 & \text{for } k \ll L^{-1} \\ \frac{55}{81} \sqrt{3} \Gamma\left(\frac{2}{3}\right) L^{-2/3} k^{-11/3} & \text{for } L^{-1} \ll k \ll \lambda^{-1} \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

In the inertial range we recover the Kolmogorov spectrum, but the power is non-zero also for small wavenumbers  $k < L^{-1}$  (a well known result in the theory of turbulence [16, 17], but which has not been taken into account in cosmological applications [10, 11]<sup>2</sup> so far).

The discontinuity of the spectrum at the peak  $k = L^{-1}$  is not real. It is a consequence of our approximations which are valid only for  $kL \ll 1$  and  $kL \gg 1$  respectively. A numerical calculation of the spectrum would smoothly turn from the  $k^2$  behavior for  $kL \ll 1$  to the  $k^{-11/3}$  for  $kL \gg 1$ . The discontinuity indicates, however, that the peak amplitude obtained by extrapolating

the asymptotic behavior right up to  $k = 1/L$  is not very reliable. Eq. (12) gives the spectrum normalisation

$$\frac{\langle v^2 \rangle}{2} = \frac{b_{ii}(0)}{2} = \int \frac{d^3 k}{(2\pi)^3} P_v(k). \quad (17)$$

Integrating Eq. (16), we obtain instead  $0.76\langle v^2 \rangle$ . This overestimation of the total energy is also due the extension of our approximations up to the peak  $k = L^{-1}$ . It indicates that the peak amplitude might be roughly a factor of 2 too high. In the following, we will neglect these discrepancies which are within the accuracy of our approximations.

We define the ratio of the total kinetic energy of turbulence to the radiation energy density (*cf.* Eq. (12)):

$$\begin{aligned} \frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} &\equiv \frac{1}{\Omega_{\text{rad}}(\eta_*)} \int_0^\infty \frac{dk}{k} \frac{d\Omega_T(k, \eta_*)}{d\log(k)} \\ &= \frac{\rho_{\text{kin}}}{\rho_*} = \frac{2\langle v^2 \rangle}{3} \simeq \int \frac{d^3 k}{6\pi^3} P_v(k). \end{aligned} \quad (18)$$

The above equation defines the turbulent energy spectrum per logarithmic unit of frequency

$$\frac{d\Omega_T(k, \eta_*)}{d\log(k)} \simeq \Omega_{\text{rad}}(\eta_*) \frac{2}{3\pi^2} k^3 P_v(k). \quad (19)$$

Usually the turbulent energy spectrum is defined as  $E(k) = k^2 P_v(k)/(2\pi^2)$  [5], in our notations we have

$$\frac{d\Omega_T(k, \eta_*)}{d\log(k)} = \Omega_{\text{rad}}(\eta_*) \frac{4}{3} k E(k). \quad (20)$$

In the Kolmogorov inertial range the energy spectrum has the form  $E(k) = C\varepsilon^{2/3} k^{-5/3}$  [5], following [10] we set the value  $C = 1$ . Comparing  $E(k)$  and the power spectrum (16) we find the relation

$$\langle v^2 \rangle \simeq 2(\varepsilon L)^{2/3}. \quad (21)$$

On large scales  $k < L^{-1}$  the above considerations on the properties of null divergence and compact support of the correlation function imply the formation of a Batchelor spectrum for the turbulent energy. The definition of the energy spectrum (19) and Eq. (11) give

$$\frac{d\Omega_T}{d\log(k)}(k \rightarrow 0) \simeq \Omega_{\text{rad}}(\eta_*) \frac{4}{9\pi} I k^5, \quad (22)$$

where  $I = \int_0^\infty dr r^4 \Gamma(r)$  is Loitsyansky's integral [5, 16, 17]. Since the correlation function has compact support, the Navier-Stokes equation requires that  $I$  be constant in time. Therefore, the turbulence is persistent on large scales and its rate of decay increases with wavenumber. Loitsyansky's integral is a measure of the angular momentum of the turbulent fluid. Its constancy is a manifestation of the conservation of angular momentum [5]. A simple estimate gives  $I \sim \langle v^2 \rangle L^5$ . Exploiting the constancy of  $I$  and the energy decay equation  $\frac{d\langle v^2 \rangle}{d\eta} \sim -\frac{\langle v^3 \rangle}{L}$ ,

<sup>2</sup> In Refs. [24, 27, 28, 29, 30, 31] the spectrum at large scales is given, but there MHD turbulence, not pure fluid turbulence is considered. Furthermore, the fact that for a divergence free vector field causality implies that the spectrum is blue,  $P_v \propto k^2$  for  $Lk \ll 1$  is only acknowledged in Ref. [24]. If we actually follow the renormalization group analysis outlined in [31], we find that at large scales,  $kL \ll 1$ , the driving force has a  $k^4$  spectrum and the stable fixed point has the critical exponents  $z = 2$  and  $\chi = -3/2$ . This can be inferred simply from causality and from the fact that the field is divergence free.

Kolmogorov has estimated the decay of the kinetic energy in freely evolving turbulence  $\langle v^2 \rangle(\eta) \propto \eta^{-10/7}$  and the growth of the correlation length  $L(\eta) \propto \eta^{2/7}$  [5, 16, 17]. In some cases these decay laws are a reasonable fit to the experimental data, nevertheless nothing tells us that they are verified for our oversimplified model of turbulence in the primordial fluid. In the following, we will simply neglect the overall decay of turbulence as far as gravitational wave production is concerned. We shall assume that turbulence appears at some time  $\eta_*$ , remains active for a short period  $\tau$  and simply disappears afterwards. We shall not model its decay in any detail.

### B. Eddy turnover and the time of duration of turbulence

The energy cascade involves a hierarchy of vortices of different sizes. To each of these structures on a given scale one can associate a characteristic velocity, the eddy velocity. Following Refs. [10, 11], we define the characteristic eddy velocity at the largest scale  $L$  by

$$\begin{aligned} v_L^2 &= \frac{3}{2} \frac{1}{\Omega_{\text{rad}}(\eta_*)} \int_{L^{-1}}^{\lambda^{-1}} \frac{dk}{k} \frac{d\Omega_T(k, \eta_*)}{d \log(k)} \\ &\simeq \langle v^2 \rangle = \frac{3}{2} \frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)}, \end{aligned} \quad (23)$$

where the second equality holds since most of the turbulent energy resides in the large wavelengths  $k \simeq L^{-1}$ . The velocity of the largest eddies,  $v_L$ , is another parameter that measures the energy stored in turbulence. For the maximal eddy velocity,  $v_L^2 = 1/3$  we have  $\frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} = \frac{2}{9}$ . In this regime, our non-relativistic approximation underestimates the kinetic energy of turbulence somewhat.

The velocity of the eddies on a length scale  $\lambda < \ell < L$  is simply

$$v_\ell = v_L \left( \frac{\ell}{L} \right)^{1/3}. \quad (24)$$

One further defines the eddy turnover time  $\tau_\ell = \ell/v_\ell$  and the turnover frequency  $\omega_\ell = 1/\tau_\ell$ .

This picture is valid only on scales  $\lambda < \ell < L$ . The principal scale of turbulence,  $L$ , defines the extent of the region over which velocities are correlated, and it is interpreted as the maximal size of turbulent eddies. On scales larger than  $L$  there is no correlated fluid motion and therefore no eddies. Similarly on scales smaller than  $\lambda$  the Reynolds number is of order unity and the fluid is not in a turbulent regime, therefore no eddies are formed, and we do not associate a turnover frequency on scales larger than  $L$  or smaller than  $\lambda$ . However, the energy distribution is not zero for  $\ell < \lambda$  and  $\ell > L$  since the eddies of a given size  $\ell$  contribute to the power spectrum

$P_v(k)$  across the full range of wavenumbers<sup>3</sup> [17].

The eddy turnover time on a given scale defines the time interval over which the eddies of that size break down into smaller ones, transferring energy from the original scale to a smaller one. This is the principle of the direct cascade. If the stirring of the fluid is continuous, fully developed and stationary turbulence is established in a time interval of the order of the turnover time on the largest scale  $\tau_L$ . Once the stirring stops, the turbulence decays away in the same time interval. If the stirring lasts for less than  $\tau_L$ , the cascade of energy develops but the turbulence is not stationary. In Ref. [10] it is argued that, from the point of view of gravitational wave production, this case can be modelled like fully developed stationary turbulence lasting for a time  $\tau_L$  (instead of for the stirring time). The assumption which goes into this model is that eddies of every size  $\ell < L$  are generated and dissipated in the time interval  $\tau_L$  (so that they make several oscillations before transferring energy to a smaller scale). Therefore, we follow [10] and define the total time of duration of turbulence as  $\tau = \max(\tau_L, \Delta\eta_*)$ . We take here  $\Delta\eta_*$  because the external stirring lasts until the phase transition is completed.

For a first order phase transition this always reduces to  $\tau = \tau_L$  [10]. From Eqs. (23) and (21) we have

$$v_L \simeq \sqrt{2}(\varepsilon L)^{1/3}, \quad \tau_L \simeq \frac{1}{\sqrt{2}} \frac{L^{2/3}}{\varepsilon^{1/3}}. \quad (25)$$

Since  $\rho_{\text{kin}} = \kappa\alpha\rho_*$ ,  $p_* = \rho_*/3$ , Eq. (13) gives

$$\varepsilon \simeq \frac{3}{4} \frac{\kappa\alpha}{\tau}. \quad (26)$$

Let us assume that the stirring lasts for much longer than the turnover time of the largest eddies. Then  $\Delta\eta_*$  should be substituted in the above equation for  $\varepsilon$ , so that

$$\tau_L = \frac{1}{\sqrt{2}} \left( \frac{4}{3} \right)^{1/3} \frac{L^{2/3}(\Delta\eta_*)^{1/3}}{(\kappa\alpha)^{1/3}}. \quad (27)$$

With this expression and using that  $L \simeq v_b(\alpha)\Delta\eta_*$ , the condition  $\Delta\eta_* \gg \tau_L$  can be rewritten in terms of the phase transition parameters,

$$\Delta\eta_* \gg \tau_L \quad \Rightarrow \quad \frac{v_b(\alpha)}{\sqrt{\alpha}} \ll \sqrt{\frac{3\kappa}{4}}. \quad (28)$$

$v_b(\alpha)$  and  $\kappa(\alpha)$  are given in Ref. [10]:

$$v_b(\alpha) = \frac{1}{1+\alpha} \left( \frac{1}{\sqrt{3}} + \sqrt{\alpha^2 + \frac{2}{3}\alpha} \right), \quad (29)$$

$$\kappa(\alpha) = \frac{1}{1+0.72\alpha} \left( 0.72\alpha + \frac{4}{27} \sqrt{\frac{3\alpha}{2}} \right). \quad (30)$$

<sup>3</sup> In the previous treatment (cf. Eq. (16)), we have neglected power on scales smaller than  $\lambda$ . The dissipation scale contains very little energy, and is too small to be relevant in the context of a cosmological treatment.

Substituting these expressions in (28) it turns out that the condition  $\Delta\eta_* \gg \tau_L$  is never verified for  $\alpha \leq 1$ . Therefore, even in the case of a strongly first order phase transition, the turbulence persists for a duration of the order of the turnover time of the largest eddies. In the following analysis we set  $\tau \simeq \tau_L$ . During this time we neglect the decay of the turbulent kinetic energy and the subsequent time dependence of the power spectrum  $P_v(k)$ . We assume that the turbulence is stationary during the time interval  $\tau_L$ , and instantaneously dissipated just afterwards.

### C. Anisotropic stress power spectrum

We now evaluate the anisotropic stress generated by the turbulence.

The energy momentum tensor of the turbulent fluid has an anisotropic stress component that induces tensor perturbations  $h_{ij}$  of the metric. The propagation equation for tensor perturbations in a radiation dominated universe is

$$\ddot{h}_{ij}(\mathbf{k}, \eta) + \frac{2}{\eta} \dot{h}_{ij}(\mathbf{k}, \eta) + k^2 h_{ij}(\mathbf{k}, \eta) = 8\pi G a^2(\eta) \Pi_{ij}(\mathbf{k}, \eta), \quad (31)$$

where  $\Pi_{ij}(\mathbf{k}, \eta)$  is the anisotropic stress. Tensor perturbations are induced on each mode  $\mathbf{k}$ . If the mode is super-horizon at the moment of generation, the last term on the left hand side of the above equation is much smaller than the damping term  $\frac{2}{\eta} \dot{h}_{ij}$  and the tensor perturbation is not yet oscillating. Once the perturbation enters the horizon, the oscillatory term can no longer be neglected and the perturbation becomes a gravitational wave in the proper sense.

If the source of the anisotropic stress has a blue power spectrum on large scales  $k < L^{-1}$ , then the anisotropic stress power spectrum is white noise (and not zero) for every  $k < L^{-1}$  as we now show. At the present time the modes  $H_0 < k < L^{-1}$  are sub-horizon. On these scales there is a background of gravitational waves coming from the primordial source, characterised by a white noise spectrum. In the case of turbulence after a phase transition, this signal has not been calculated in previous analysis.

The energy momentum tensor of the fluid is

$$T_{\mu\nu} = (\rho + p) v_\mu v_\nu + p g_{\mu\nu}. \quad (32)$$

We extract the spatial, transverse and traceless part which is relevant for tensor perturbations. The turbulence is active on sub-horizon scales  $k \gg 1/\eta_*$  and for a time much shorter than one Hubble time. For this short period of time, the expansion of the universe can be neglected, and we may assume a Minkowski background. In Eq. (32) only the second order term  $T_{ij}(\mathbf{x}, \eta_*) = 4/3 \rho_* v_i(\mathbf{x}, \eta_*) v_j(\mathbf{x}, \eta_*)$  generates tensor perturbations [10]. We have substituted  $p_* = \rho_*/3$  since the phase transition takes place during the radiation

dominated era. In  $\mathbf{k}$ -space, the tensor anisotropic stress is given by  $\Pi_{ij}(\mathbf{k}) = (P_{il} P_{jm} - 1/2 P_{ij} P_{lm}) T_{lm}(\mathbf{k})$  where  $P_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j$  is the transverse traceless projector, and  $T_{ij}(\mathbf{k}, \eta_*) = \frac{4}{3} \rho_* \int \frac{d^3 q}{(2\pi)^3} v_i(\mathbf{q}, \eta_*) v_j(\mathbf{k} - \mathbf{q}, \eta_*)$ , up to an irrelevant trace which vanishes in the projection. The anisotropic stress power spectrum  $\langle \Pi_{ij}(\mathbf{k}, \eta_*) \Pi_{ij}^*(\mathbf{q}, \eta_*) \rangle$  has already been calculated in Ref. [10]: using Wick's theorem to reduce the four point spectral function of the velocity, one obtains

$$\langle \Pi_{ij}(\mathbf{k}, \eta_*) \Pi_{ij}^*(\mathbf{q}, \eta_*) \rangle = \left( \frac{4\rho_*}{3} \right)^2 \delta(\mathbf{k} - \mathbf{q}) \Pi_v(k, \eta_*), \quad (33)$$

$$\Pi_v(k, \eta_*) = \int d^3 p P_v(p, \eta_*) P_v(|\mathbf{k} - \mathbf{p}|, \eta_*) (1 + \gamma^2)(1 + \beta^2), \quad (34)$$

with  $\gamma = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$ ,  $\beta = \widehat{\mathbf{k} \cdot \mathbf{k} - \mathbf{p}}$ . In principle, we are not allowed to use Wick's theorem, the turbulent velocity field being not at all Gaussian. We invoke it here for simplicity and to obtain an expression for the four-point function. We hope that even if not correct in detail, this gives the right order of magnitude for the four point function. Reducing the four point spectral function to the power spectrum by means of Wick's theorem is sometimes used as a closure method also in more detailed studies of turbulence [17].

The anisotropic stress power spectrum is explicitly calculated at the time of the phase transition  $\eta_*$ . The time evolution equation for the turbulent power spectrum is highly non linear and involves the three point function [5]. As already mentioned in Section II, in our approximate approach to the evaluation of gravitational waves generated from the turbulence, we ignore the detailed time dependence of the anisotropic stress, since it is not very relevant for the resulting gravitational waves which are given by a time integral of the source. This however does not mean that we are treating the turbulence as a coherent source, as will become clear in the next section.

The integral in (34) cannot be done exactly, but good approximations have already been derived in Ref. [10] for small scales  $k > L^{-1}$ , and in Refs. [18, 20] for large scales  $k < L^{-1}$ . With Eqs. (16) and (18), we find<sup>4</sup>

$$\Pi_v(k) \simeq \frac{27}{8} \pi^4 \left( \frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} \right)^2 \begin{cases} \frac{2}{13} L^3 & \text{for } k < L^{-1} \\ L^{-2/3} k^{-11/3} & \text{for } L^{-1} < k < \lambda^{-1} \\ 0 & \text{otherwise.} \end{cases} \quad (35)$$

<sup>4</sup> The analytical approximations used in the above mentioned references are not entirely correct, because of the bounds that have been used in the angular integration.  $\gamma = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$  is not varying from  $-1$  to  $1$  but its bounds depend on  $\mathbf{k}$  and  $\mathbf{p}$ . We have checked that using the correct bounds does not significantly modify the approximated final result.

As anticipated, the tensor source power spectrum is white noise on large scales. However, because of the convolution, the largest contribution to  $\Pi_v(k)$  on these scales does not come from the velocity power spectrum on the same scales, but from the value of the velocity power spectrum at the maximum,  $k \simeq L^{-1}$ . The discontinuity of the spectrum at  $k = L^{-1}$  is not physical, it is due to the approximations we used in evaluating integral (34).

#### D. Generation of gravitational waves

We now determine the spectrum of gravitational waves which are induced by the turbulent motion of the primordial fluid. The turbulent velocity is a random variable, but within Richardson's model of the energy cascade it is possible to associate a frequency to the fluid motions at each scale, since the fluid motions are vortical. This 'oscillatory' behaviour is peculiar to each realisation of turbulence, and is lost in the statistical average. The velocity power spectrum  $P_v(k)$  is not oscillating in time. First of all, the Fourier transform decomposes the velocity field into a hierarchy of waves, not of eddies. Moreover, a statistical model cannot reproduce the detailed dynamics of individual vortex events. However, the 'oscillatory' behaviour of the source is relevant for the generation of the gravitational wave background.

In order to account for these oscillations which are present in turbulent motion on scales smaller than  $L$ , we model the random velocity of the turbulent fluid heuristically as

$$v_i(\mathbf{k}, \eta) = \begin{cases} v_i(\mathbf{k}) & \text{for } k < L^{-1} \\ v_i(\mathbf{k})e^{i\omega_k\eta} & \text{for } L^{-1} < k < \lambda^{-1}, \end{cases} \quad (36)$$

where we do not associate a turnover frequency to scales larger than the correlation scale. Here  $\omega_k$  is the frequency of an eddy with size  $\ell < L$ ,

$$\begin{aligned} \omega_k &= v_\ell/\ell = (v_L/L^{1/3})\ell^{-2/3} \\ &= (v_L/L^{1/3})k^{2/3} \ll k, \quad k = 1/\ell. \end{aligned} \quad (37)$$

The energy momentum tensor of the source in  $\mathbf{k}$  space involves a convolution (*cf.* section III C):

$$T_{ij}(\mathbf{k}, \eta) = \frac{4}{3}\rho(\eta) \int \frac{d^3q}{(2\pi)^3} v_i(\mathbf{q}) v_j(\mathbf{k} - \mathbf{q}) e^{i\omega_q\eta} e^{i\omega_{|\mathbf{k}-\mathbf{q}|}\eta}. \quad (38)$$

The convolution entails that the energy momentum tensor as a random variable is oscillating also on large scales  $k < L^{-1}$ , contrary to the random velocity field. At these scales the amplitude is dominated by the contributions from  $q = 1/L$  (*cf.* Eq. (35)). We therefore expect the energy momentum tensor to oscillate with the smallest frequency  $\omega_L$ . This can be approximatively justified by the following reasoning. Since we concentrate on the case of turbulence generated by a first order phase transition, the source is active for a time  $\tau_L$ . Therefore, the main contribution to the convolution integral comes from frequencies such that  $\omega_q\tau_L \sim 1$ ,  $\omega_{|\mathbf{k}-\mathbf{q}|}\tau_L \sim 1$ . If  $k < L^{-1}$  and

$q < L^{-1}$  the frequencies are both zero, but for  $k < L^{-1}$  and  $q > L^{-1}$  one can approximate  $\omega_{|\mathbf{k}-\mathbf{q}|} \sim \omega_q$  so that the frequency is  $2\omega_q$ . The same argument for smaller scales  $k > L^{-1}$  gives the frequency  $2\omega_k$ . We can therefore model the time dependence of the random anisotropic stress tensor by

$$\Pi_{ij}(\mathbf{k}, \eta) = \begin{cases} \Pi_{ij}(\mathbf{k})e^{2i\omega_L\eta} & \text{for } k \leq L^{-1} \\ \Pi_{ij}(\mathbf{k})e^{2i\omega_k\eta} & \text{for } L^{-1} \leq k < \lambda^{-1}. \end{cases} \quad (39)$$

With this source the gravitational wave equation becomes the equation of a forced harmonic oscillator with damping,

$$\begin{aligned} \ddot{h}_{ij}(\mathbf{k}, \eta) + \frac{2}{\eta}\dot{h}_{ij}(\mathbf{k}, \eta) + k^2 h_{ij}(\mathbf{k}, \eta) &= \\ = 8\pi G a^2(\eta) \Pi_{ij}(\mathbf{k}) e^{2i\omega\eta} \Theta(\eta - \eta_{\text{in}}) \Theta(\eta_{\text{fin}} - \eta). \end{aligned} \quad (40)$$

Here the source frequency is generically denoted  $2\omega$ , and  $\eta_{\text{in}}$  and  $\eta_{\text{fin}}$  are the initial and final times of action of the source,  $\eta_{\text{fin}} - \eta_{\text{in}} = \tau_L$ . The Heaviside functions  $\Theta$  are inserted to limit the source to the interval  $\eta_{\text{in}} < \eta < \eta_{\text{fin}}$ . The source is active for a time much shorter than one Hubble time:  $\tau_L \ll \eta_*$ . Therefore we can neglect the expansion of the universe during the relatively short period during which the source is active,

$$\begin{aligned} \ddot{h}_{ij}(\mathbf{k}, \eta) + k^2 h_{ij}(\mathbf{k}, \eta) &= \\ = 8\pi G a_*^2 \Pi_{ij}(\mathbf{k}) e^{2i\omega\eta} \Theta(\eta - \eta_{\text{in}}) \Theta(\eta_{\text{fin}} - \eta). \end{aligned} \quad (41)$$

The solutions to the homogeneous equation without damping are simply  $\exp(\pm ik\eta)$ , and with the Green's function method one finds the inhomogeneous solution for  $\eta \leq \eta_{\text{fin}}$ ,

$$\begin{aligned} h_{ij}(\mathbf{k}, \eta \leq \eta_{\text{fin}}) &= \frac{4\pi G a_*^2 \Pi_{ij}(\mathbf{k})}{k} \times \\ &\left[ e^{ik\eta} \frac{e^{i(2\omega-k)\eta} - e^{i(2\omega-k)\eta_{\text{in}}}}{k - 2\omega} + \right. \\ &\left. e^{-ik\eta} \frac{e^{i(2\omega+k)\eta} - e^{i(2\omega+k)\eta_{\text{in}}}}{k + 2\omega} \right]. \end{aligned} \quad (42)$$

Note that the singularity at  $k = 2\omega$  of the first term is removable. For  $\eta_{\text{in}} < \eta < \eta_{\text{fin}}$  the time dependence of gravitational waves with wave number  $\mathbf{k}$  is given by the superposition of the source frequency,  $2\omega$ , and of the source wavenumber  $k$ . However, once the source ceases to be active, the gravitational wave propagates freely at the speed of light. In order to find a solution for  $\eta \gg \eta_{\text{fin}}$ , we need to match Eq. (42) at  $\eta_{\text{fin}}$  with a generic solution of the homogeneous propagation equation

$$\ddot{h}_{ij}(\mathbf{k}, \eta) + \frac{2}{\eta}\dot{h}_{ij}(\mathbf{k}, \eta) + k^2 h_{ij}(\mathbf{k}, \eta) = 0. \quad (43)$$

Now the expansion of the universe can no longer be neglected, and the homogeneous solutions in this case are

$\exp(\pm ik\eta)/\eta$ . The matching procedure leads to

$$h_{ij}(\mathbf{k}, \eta) = \frac{4\pi G a_*^2 \eta_* \Pi_{ij}(\mathbf{k})}{k} \times \left[ \frac{e^{ik\eta}}{\eta} \frac{e^{i(2\omega-k)\eta_{\text{fin}}} - e^{i(2\omega-k)\eta_{\text{in}}}}{k-2\omega} + \frac{e^{-ik\eta}}{\eta} \frac{e^{i(2\omega+k)\eta_{\text{fin}}} - e^{i(2\omega+k)\eta_{\text{in}}}}{k+2\omega} \right] \quad (44)$$

(where we have substituted  $\eta_{\text{fin}}$  by  $\eta_*$  in the amplitude). For  $\eta \gg \eta_*$  the gravitational wave propagates freely with the correct dispersion relation: its frequency is just  $\pm k$ , and the time dependence  $\eta^{-1}$  accounts for the expansion of the universe. The frequency  $2\omega$  of the source only enters in the amplitude.

We want to evaluate the gravitational wave energy density today, normalised to the critical density  $\rho_c$ . In real space the gravitational wave energy density parameter is [21]

$$\Omega_G = \frac{\rho_G}{\rho_c} = \frac{\langle \dot{h}_{ij} \dot{h}^{ij} \rangle}{16\pi G \rho_c a^2}. \quad (45)$$

Here  $\langle \dots \rangle$  denotes ensemble average and, at the same time, time averaging over several periods of oscillation. The factor  $1/a^2$  comes, since the overdot denotes the derivative w.r.t. conformal time.

We define the gravitational wave power spectrum by

$$\langle \dot{h}_{ij}(\mathbf{k}, \eta) \dot{h}_{ij}^*(\mathbf{q}, \eta) \rangle = \delta(\mathbf{k} - \mathbf{q}) |\dot{h}|^2(k, \eta). \quad (46)$$

Fourier transforming the above relation for  $\Omega_G$  one defines the gravitational wave energy density per logarithmic unit of frequency:

$$\Omega_G = \int \frac{dk}{k} \frac{d\Omega_G(k)}{d\log(k)}, \quad (47)$$

with

$$\frac{d\Omega_G(k)}{d\log(k)} = \frac{k^3 |\dot{h}|^2}{4(2\pi)^6 G \rho_c a^2}. \quad (48)$$

This is the quantity we are interested in. To proceed we need to find an expression for the spectrum (46) at late times (today) and on sub-horizon scales  $k\eta \gg 1$ . From solution (44) with  $k\eta \gg 1$  one has, up to an irrelevant overall phase

$$\dot{h}_{ij}(\mathbf{k}, \eta) = \frac{4\pi G a_*^2 \eta_* \Pi_{ij}(\mathbf{k})}{\eta} \times \left[ A e^{ik(\eta-\eta_{\text{in}})} - B e^{-ik(\eta-\eta_{\text{in}})} \right], \quad (49)$$

where  $A = \frac{e^{i(2\omega-k)\tau_L} - 1}{k-2\omega}$  and  $B = \frac{e^{i(2\omega+k)\tau_L} - 1}{k+2\omega}$ . Given the anisotropic stress power spectrum (33), the gravitational wave power spectrum takes the form

$$|\dot{h}|^2(k, \eta) = (4\pi G)^2 a_*^4 \eta_*^2 \left( \frac{4\rho_*}{3} \right)^2 \Pi_v(k, \eta_*) \frac{|A|^2 + |B|^2}{\eta^2}. \quad (50)$$

We now use Eqs. (48) and (50) to evaluate the gravitational wave energy density spectrum. The main spectral dependence comes from the anisotropic stress spectrum (35), which we write in the form

$$k^3 \Pi_v(k) \simeq \frac{27}{8} \pi^4 \left( \frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} \right)^2 \begin{cases} \frac{2}{13} x^3 & \text{for } 0 < x < 1 \\ x^{-2/3} & \text{for } 1 < x < \frac{L}{\lambda} \\ 0 & \text{otherwise,} \end{cases} \quad (51)$$

where we have defined the variable  $x = kL$ . Moreover, the value of  $\omega$  in the amplitudes  $A$  and  $B$  depends on  $k$ . From Eq. (39), we have  $\omega = \omega_L$  for scales  $k < L^{-1}$ , and  $\omega = \omega_k \propto k^{2/3}$  for scales  $k > L^{-1}$ . Using the expression for the scale factor in the radiation dominated era  $a = H_0 \sqrt{\Omega_{\text{rad}}} \eta$ , we finally obtain the energy density spectrum,

$$\frac{d\Omega_G(k, \eta)}{d\log(k)} \simeq \frac{9}{64\pi} (\mathcal{H}_* L)^2 \left( \frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} \right)^2 \frac{\Omega_{\text{rad}}}{a^4(\eta)} \begin{cases} \frac{2}{13} \left[ \frac{|e^{2i(1-\frac{x}{2v_L})} - 1|^2}{(x-2v_L)^2} + \frac{|e^{2i(1+\frac{x}{2v_L})} - 1|^2}{(x+2v_L)^2} \right] x^3 & \text{for } 0 < x < 1, \\ \left[ \frac{|e^{2i(x^{2/3}-\frac{x}{2v_L})} - 1|^2}{(x-2v_L x^{2/3})^2} + \frac{|e^{2i(x^{2/3}+\frac{x}{2v_L})} - 1|^2}{(x+2v_L x^{2/3})^2} \right] x^{-2/3} & \text{for } 1 < x < \frac{L}{\lambda} \\ 0 & \text{otherwise.} \end{cases} \quad (52)$$

Once the source has decayed, the gravity wave energy density evolves like radiation, as it should. The energy

spectrum today is simply given by the above expression evaluated at  $\eta_0$  with  $a(\eta_0) = 1$ . We obtain a different



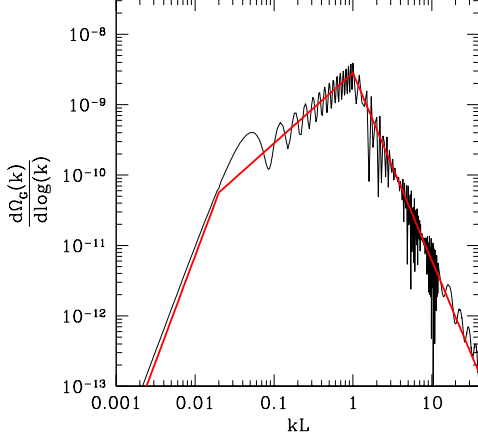


FIG. 1: The gravitational wave energy density today, normalised to  $(\mathcal{H}_* L)^2 \Omega_{\text{rad}}$ , as a function of  $x = kL$  is shown. We have chosen  $v_L^2 = \frac{3}{2} \frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} = 10^{-4}$ . The black, oscillating line shows the exact amplitude given in Eq. (52). The red, straight line shows the approximated spectrum given in Eq. (53), normalised to the same quantity  $(\mathcal{H}_* L)^2 \Omega_{\text{rad}}$ . The factor  $2/13$  appearing in Eq. (52) has been neglected in the plot since it generates an unphysical discontinuity.

spectral dependence whether the largest eddy velocity  $v_L$  is below or above  $1/2$ . For  $v_L \leq 1/2$ , the characteristic frequency  $2\omega_L$  is less than the scale  $L^{-1}$  at which the anisotropic stress power spectrum peaks (51). The gravitational wave energy spectrum still peaks at  $L^{-1}$ , but it changes slope at  $k = 2\omega_L$ . An approximate solution for the energy spectrum in the case  $v_L \leq 1/2$  is

$$\frac{d\Omega_G(k, \eta_0)}{d \log(k)} \simeq \frac{9}{32\pi} (\mathcal{H}_* L)^2 \left( \frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} \right)^2 \Omega_{\text{rad}} \times \begin{cases} x^3/v_L^2 & \text{for } 0 < x < 2v_L \\ 4x & \text{for } 2v_L < x < 1 \\ 4x^{-8/3} & \text{for } 1 < x < \frac{L}{\lambda} \\ 0 & \text{otherwise.} \end{cases} \quad (53)$$

The value of the energy spectrum normalised to the parameters  $(\mathcal{H}_* L)^2 \Omega_{\text{rad}}$  is shown in Fig. 1 for  $v_L = 0.01$  and in Fig. 2 for  $v_L = 1/2$ . We remind that  $\frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} = \frac{2}{3} v_L^2$  (cf. Eq. (23)).

If  $v_L \geq 1/2$ , then  $2\omega_k > 2\omega_L > L^{-1}$ . The velocity of the largest eddies is bounded by the speed of sound  $v_L \leq 1/\sqrt{3}$  [10, 15]. Also in this case, the energy spectrum peaks at the frequency  $L^{-1}$ , but because of the particular form of the amplitude it changes slope at  $k = (2v_L)^3/L$ .

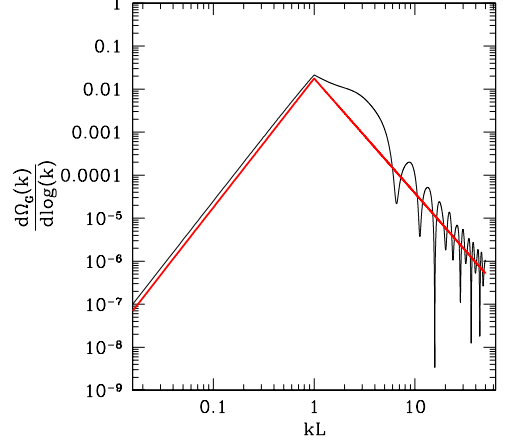


FIG. 2: Like Fig. 1, but with  $v_L = 1/2$ , hence  $\frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} = 1/6$ .

An approximate solution for  $v_L \geq 1/2$  is

$$\frac{d\Omega_G(k, \eta_0)}{d \log(k)} \simeq \frac{9}{32\pi} (\mathcal{H}_* L)^2 \left( \frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} \right)^2 \Omega_{\text{rad}} \times \begin{cases} x^3/v_L^2 & \text{for } 0 < x < 1 \\ x^{-2}/v_L^2 & \text{for } 1 < x < (2v_L)^3 \\ 4x^{-8/3} & \text{for } (2v_L)^3 < x < \frac{L}{\lambda} \\ 0 & \text{otherwise.} \end{cases} \quad (54)$$

This result is plotted in Fig. 3 for the maximal value  $v_L = 1/\sqrt{3}$ ,  $\frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} = \frac{2}{9}$ . Eqs. (53) and (54) expressed in terms of the parameters which describe the phase transition (see Refs. [11, 12]), can be found in Appendix B.

The maximal gravitational wave signal is obtained for the highest possible value of the eddy velocity,  $v_L$ . Given that  $L \simeq v_b(\alpha) \Delta\eta_*$ , we obtain a peak frequency of

$$k_L \equiv L^{-1} \simeq 8 \times 10^{-3} \frac{1}{v_b(\alpha)} \frac{1}{\Delta\eta_* \mathcal{H}_*} \frac{T_*}{100 \text{ GeV}} \text{ mHz} \quad (55)$$

(we normalise it to the electroweak phase transition temperature). Moreover, the phase transition has to be strongly first order, so that the bubbles expand at the speed of light. If  $v_b(\alpha) \simeq 1$  and  $\Delta\eta_* \mathcal{H}_* \simeq 10^{-2}$  as argued before, the spectrum peaks at a frequency of about mHz. For the maximal eddy velocity of  $v_L = 1/\sqrt{3}$ , we obtain the amplitude at the peak frequency

$$\left. \frac{d\Omega_G(k, \eta_0)}{d \log(k)} \right|_{k_L} \simeq \frac{1}{24\pi} \times 10^{-4} \Omega_{\text{rad}} \simeq 10^{-10}. \quad (56)$$

The total energy density in gravitational waves for  $1/2 \leq$

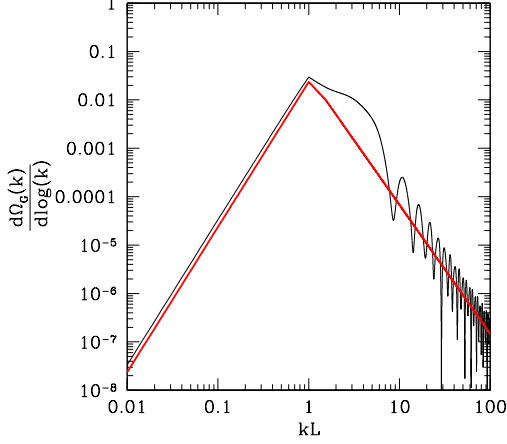


FIG. 3: The figure shows the gravitational wave energy density today, normalised to  $(\mathcal{H}_* L)^2 \Omega_{\text{rad}}$ , as a function of  $x = kL$ , for  $v_L = 1/\sqrt{3}$ , hence  $\frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} = \frac{2}{9}$ . The red, straight line is the approximation given in Eq. (54).

$v_L \leq 1/\sqrt{3}$  is given by the integral of Eq. (54),

$$\begin{aligned} \Omega_G &= \int_0^\infty \frac{dk}{k} \frac{d\Omega_G(k, \eta_0)}{d\log(k)} \\ &\simeq \frac{5}{32\pi} \left(\frac{L}{\eta_*}\right)^2 \frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} \Omega_{\text{rad}}. \end{aligned} \quad (57)$$

For the maximal value  $v_L = 1/\sqrt{3}$  we have therefore

$$\begin{aligned} \Omega_G &\simeq \frac{5}{144\pi} \left(\frac{L}{\eta_*}\right)^2 \Omega_{\text{rad}} \\ &\simeq 0.01 v_b^2(\alpha) \left(\frac{\Delta\eta_*}{\eta_*}\right)^2 \Omega_{\text{rad}}. \end{aligned} \quad (58)$$

We find that the turbulent phase arising after a strongly first order phase transition produces a background of gravitational waves which is in principle detectable by the space interferometer LISA. However, the gravity wave intensity obtained here depends strongly on the peak amplitude of the turbulent spectrum, which is rather uncertain. Therefore, also the amount of gravitational waves generated is probably not very accurate.

Our result is in agreement with Ref. [11], with a few differences. Since we account for the natural dispersion relation of gravitational waves  $\omega = k$ , we obtain that the spectrum peaks at the frequency  $1/L$ , and not at the turnover frequency  $\omega_L$  as found in [9, 10, 11]. The two differ by a factor  $v_L$ . In our case the characteristic size of the eddies determines the frequency of the induced gravitational waves, while the eddy velocity enters only in the overall amplitude (see Appendix A). We therefore also have a different slope of the spectrum for  $k > L^{-1}$  than Refs. [9, 10, 11]. Moreover, we find that the gravitational

wave spectrum is not reduced only to the ‘Kolmogorov’ part, but it continues to wavelengths larger than  $1/L$  as shown in Figs 1 to 3. For a strongly first order phase transition, the amplitude of the signal at mHz frequencies derived here is of the same order of magnitude as found in [11], within the range of our analytical approximations.

## IV. GRAVITATIONAL WAVES FROM MAGNETIC FIELDS

### A. The magnetic field power spectrum

We now analyse the production of gravitational waves by a primordial magnetic field. In order to make a direct comparison with the turbulence case, we consider here only causally created magnetic fields. Typically, like turbulence, magnetic fields can be generated during a phase transition (note however that in the case of the electroweak phase transition, magnetic seed fields can form even if the phase transition is second order [8]).

The magnetic field power spectrum on large scales is determined along the same lines as the turbulence power spectrum in Section II. The field is divergence-free and it must be uncorrelated above a certain scale, therefore the slope has to be  $k^n$  with  $n \geq 2$  an even integer. This slope continues up to the correlation scale. If the magnetic field is simply constant on scales smaller than the correlation scale the power spectrum goes to zero steeply on these scales (in [24] for example it is modeled as  $\exp(-(k/k_c)^4)$ ). This is normally assumed to be the initial spectrum. However as the system evolves, the interactions of the magnetic field with the cosmic fluid modify the spectrum at small scales, turning it into a power law (for example, into the Iroshnikov Kraichnan spectrum,  $E_B(k) = C(\varepsilon v_A)^{1/2} k^{-3/2}$  with  $v_A$  the Alfvén speed, see [10, 11, 24, 25]). To stay general, we make the ansatz

$$\langle B_i(\mathbf{k}) B_j^*(\mathbf{q}) \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{q}) (\delta_{ij} - \hat{k}_i \hat{k}_j) P_B(k), \quad (59)$$

$$P_B(k) = \begin{cases} C_B k^2 & \text{for } 0 < k < L^{-1} \\ C_B L^{\alpha-2} k^\alpha & \text{for } L^{-1} < k < \lambda^{-1} \\ 0 & \text{otherwise,} \end{cases} \quad (60)$$

with  $\alpha < -3$ , so that the integrated energy is dominated by the contribution from the correlation scale  $L$ .  $L$  and  $\lambda$  are not necessarily the same scales as for turbulence but are the magnetic correlation and dissipation scales.  $C_B$  is as yet undetermined constant.

Once the magnetic field is generated, it is not quickly dissipated like turbulence. Because of the high conductivity of the cosmic plasma there is negligible Ohmic dissipation and the magnetic flux is conserved. During the early stages of evolution of the universe both, the kinetic and magnetic Reynolds numbers are very high, and magnetohydrodynamical turbulence is generated. The magnetic field decay has been analysed in this case both ana-

lytically [26, 27, 28, 29] and numerically [24, 25], and different scaling laws are obtained depending on the treatment of the problem. In the following we keep the scaling exponent as a free parameter, since it will not be very relevant for the final result. However, all papers agree on the fact that the power spectrum is persistent on large scales, *i.e.* that  $C_B$  is constant in time.

After electron-positron annihilation the neutrinos are already decoupled and the photon mean free path increases suddenly. Then the system is no longer turbulent but dissipative due to radiation viscosity, and the magnetic field evolution changes [22, 23, 25].

The main point for our analysis is that in the process of generation of gravitational waves, one needs in principle to account for the evolution of the magnetic spectrum. This can influence the final result, since the magnetic field is acting as a source of gravitational waves for a cosmologically long time. The situation is different from the case of turbulence. This evolution has not been taken into account in [18], where a fixed correlation length given by the size of the horizon at the time of generation has been assumed.

As for the slopes of the magnetic spectrum (60), we make a general Ansatz also for its time evolution. The evolution is mainly determined by three processes: the expansion of the universe, the growth of the correlation length and the decay of the energy density. To eliminate the time dependence by redshifting we work with the comoving magnetic energy density

$$\langle B^2 \rangle = \langle B^2(\eta) \rangle a^4(\eta). \quad (61)$$

However, as we shall see below,  $\langle B^2 \rangle$  is still time dependent due to dissipation of energy.

As in the case of the turbulence, we normalise the spectrum (60) in terms of the total magnetic energy density divided by the radiation energy density:

$$\begin{aligned} \frac{\Omega_B(\eta)}{\Omega_{\text{rad}}(\eta)} &= \frac{1}{\rho_{\text{rad}}(\eta_0)} \frac{\langle B^2 \rangle}{8\pi} \\ &= \frac{1}{\rho_{\text{rad}}(\eta_0)} \frac{1}{(2\pi)^3} \int_0^\infty \frac{dk}{k} k^3 P_B(k) \\ &\equiv \frac{1}{\Omega_{\text{rad}}} \int_0^\infty \frac{dk}{k} \frac{d\Omega_B}{d \log k} \end{aligned} \quad (62)$$

which gives

$$\begin{aligned} C_B &\simeq \frac{(2\pi)^3}{8\pi} \frac{5(\alpha+3)}{\alpha-2} \langle B^2 \rangle L^5 \\ &= (2\pi)^3 \frac{5(\alpha+3)}{\alpha-2} \Omega_B \rho_c L^5. \end{aligned} \quad (63)$$

Here we have defined the magnetic field energy density parameter  $\Omega_B$  which corresponds to the magnetic field energy density scaled to today via Eq. (61).  $\Omega_B$  is itself a function of time due to the dissipation of energy.

Let us now study the time dependence of  $L$  and  $\langle B^2 \rangle$ . We model the comoving correlation length by

$$L(\eta) = L_* \left( \frac{\eta}{\eta_*} \right)^\gamma, \quad (64)$$

with  $0 < \gamma < 1$  so that  $L(\eta)$  never overcomes the horizon at any instant  $\eta$ .  $L_*$  denotes the correlation length at the epoch  $\eta_*$  of creation of the magnetic field. The time dependence of the dissipation scale  $\lambda$  does not play a significant role in our analysis, so we do not consider it. For our purposes, it is enough to have in mind that  $\lambda$  is also a function growing with time. In order to find the time evolution of the magnetic energy density, we make an analogy: we assume that from the MHD equation it is possible to define a conserved quantity analogous to the Loitsyansky's integral. We impose simply

$$\langle B^2 \rangle L^5 = \text{constant in time}, \quad (65)$$

with this the large scale part of the spectrum is persistent (see Eq. (63) above). This is equivalent to the constancy of  $C_B$ , which has been observed in simulations [24, 25]. With this the comoving magnetic energy scales as

$$\langle B^2 \rangle(\eta) = \langle B^2 \rangle(\eta_*) \left( \frac{\eta_*}{\eta} \right)^{5\gamma}. \quad (66)$$

If we were to adopt the same decay laws as in turbulence, one would find the Kolmogorov decay laws  $L(\eta) \propto \eta^{2/7}$ ,  $\langle B^2 \rangle(\eta) \propto \eta^{-10/7}$ . Equivalent scaling laws are obtained from the arguments of self similarity used in [24, 26], and of selective decay used in [25, 28]. The main difference is that most of these works consider a white noise spectrum for the magnetic field at large scales, resulting in a slower decay of the energy and a more substantial growth of the correlation length. They find Saffman's law for fluid dynamics. However, as demonstrated in Section II, causally generated, divergence-free vector fields in the early universe necessarily lead to the formation of a Batchelor spectrum. This is valid for both, turbulence and magnetic fields.

## B. Power spectrum of the anisotropic stress

From the (comoving) magnetic field stress tensor

$$T_{ij}^{(B)} = \frac{1}{4\pi} \left[ \frac{1}{2} B^2 g_{ij} - B_i B_j \right] \quad (67)$$

one defines again the anisotropic stress as  $\Pi_{ij}(\mathbf{k}) = (P_{il}P_{jm} - 1/2 P_{ij}P_{lm})T_{lm}(\mathbf{k})$ . One then calculates the comoving anisotropic stress power spectrum in the usual way, see *e.g.* [19], using Wick's theorem to reduce 4-point to 2-point correlators under the assumption of Gaussianity (which again is most probably not strictly valid but can be expected to give the right order of magnitude). Setting

$$\langle \Pi_{ij}(\mathbf{k}, \eta) \Pi_{ij}^*(\mathbf{q}, \eta) \rangle \equiv \Pi_B(k, \eta) \delta(\mathbf{k} - \mathbf{q}) \quad (68)$$

one obtains, exactly like in Eq. (34),

$$\Pi_B(k, \eta) = \int d^3p P_B(p, \eta) P_B(|\mathbf{k} - \mathbf{p}|, \eta) (1 + \gamma^2)(1 + \beta^2). \quad (69)$$

A good approximation to this convolution integral is

$$\Pi_B(k, \eta) \simeq C_B^2 \begin{cases} A_1^2 L^{-7}(\eta) & \text{for } 0 < k < L^{-1} \\ A_2^2 L^{\alpha-7}(\eta) k^\alpha & \text{for } L^{-1} < k < \lambda^{-1} \\ 0 & \text{otherwise,} \end{cases} \quad (70)$$

with  $A_1 = \sqrt{\frac{2\alpha-4}{7(2\alpha+3)}}$  and  $A_2 = \sqrt{\frac{\alpha-2}{5(2\alpha+3)}}$ .

The time evolutions of the magnetic energy density and of the correlation scale are deterministic. The same holds for the evolution of the anisotropic stress power spectrum. We therefore may treat magnetic fields as a coherent source, *i.e.* a source with deterministic time dependence,

$$\langle \Pi_{ij}(\mathbf{k}, \eta_1) \Pi_{ij}^*(\mathbf{q}, \eta_2) \rangle = f(\eta_1) f(\eta_2) \langle \Pi_{ij}(\mathbf{k}, \eta_{\text{in}}) \Pi_{ij}^*(\mathbf{q}, \eta_{\text{in}}) \rangle \delta(\mathbf{k} - \mathbf{q}), \quad (71)$$

( $f(\eta)$  denotes here an arbitrary function). Using  $\Pi_B(\mathbf{k}, \eta) = f^2(\eta) \Pi_B(\mathbf{k}, \eta_{\text{in}})$ , we find

$$\langle \Pi_{ij}(\mathbf{k}, \eta_1) \Pi_{ij}^*(\mathbf{q}, \eta_2) \rangle = \sqrt{\Pi_B(k, \eta_1)} \sqrt{\Pi_B(k, \eta_2)} \delta(\mathbf{k} - \mathbf{q}). \quad (72)$$

### C. Generation of gravitational waves

In the case of a coherent source, one can write the evolution equation (31) directly for the gravitational wave power spectrum. To see this, we first write it for the stochastic comoving anisotropic stress of the magnetic field

$$\ddot{h}_{ij} + \frac{2}{\eta} \dot{h}_{ij} + k^2 h_{ij} = 8\pi G \frac{\Pi_{ij}}{a^2}. \quad (73)$$

The solution to this equation with initial condition  $h_{ij}(\eta_*) = 0$  is

$$h_{ij}(\mathbf{k}, \eta) = 8\pi G \int_{\eta_*}^{\eta} d\zeta \frac{\Pi_{ij}(\mathbf{k}, \zeta)}{a^2(\zeta)} \mathcal{G}(\eta, \zeta, \mathbf{k}), \quad (74)$$

where  $\mathcal{G}(\eta, \zeta, \mathbf{k})$  is the corresponding Green function. The gravitational wave energy power spectrum becomes

$$\begin{aligned} \langle \dot{h}_{ij}(\mathbf{k}, \eta) \dot{h}_{ij}^*(\mathbf{q}, \eta) \rangle &= (8\pi G)^2 \frac{d}{d\eta} \int_{\eta_*}^{\eta} d\zeta \frac{\mathcal{G}(\eta, \zeta, \mathbf{k})}{a^2(\zeta)} \\ &\times \frac{d}{d\eta} \int_{\eta_*}^{\eta} d\xi \frac{\mathcal{G}^*(\eta, \xi, \mathbf{q})}{a^2(\xi)} \langle \Pi_{ij}(\mathbf{k}, \zeta) \Pi_{ij}^*(\mathbf{q}, \xi) \rangle. \end{aligned} \quad (75)$$

For a coherent source, using (72), we find

$$\begin{aligned} \langle \dot{h}_{ij}(\mathbf{k}, \eta) \dot{h}_{ij}^*(\mathbf{q}, \eta) \rangle &= |\dot{h}|^2(k, \eta) \delta(\mathbf{k} - \mathbf{q}) = \\ &= \delta(\mathbf{k} - \mathbf{q}) \left| 8\pi G \frac{d}{d\eta} \int_{\eta_*}^{\eta} d\zeta \frac{\sqrt{\Pi_B(k, \zeta)}}{a^2(\zeta)} \mathcal{G}(\eta, \zeta, \mathbf{k}) \right|^2 \end{aligned} \quad (76)$$

where we can identify

$$h(k, \eta) = 8\pi G \int_{\eta_*}^{\eta} d\zeta \frac{\sqrt{\Pi_B(k, \zeta)}}{a^2(\zeta)} \mathcal{G}(\eta, \zeta, \mathbf{k}). \quad (77)$$

Since  $\dot{h}_{ij}(\mathbf{k}, \eta)$  and  $h(k, \eta)$  share the same Green's function,  $h(k, \eta)$  satisfies equation (73), with source term  $8\pi G \sqrt{\Pi_B(k, \eta)}/a^2$ . In the radiation era this is

$$\ddot{h} + \frac{2}{\eta} \dot{h} + k^2 h = \frac{8\pi G}{H_0^2 \Omega_{\text{rad}} \eta^2} \sqrt{\Pi_B(k, \eta)}. \quad (78)$$

With the power spectrum (70), the solution is

$$\begin{aligned} h(k, \eta) &= \frac{4\pi G}{H_0^2 \Omega_{\text{rad}}} \frac{i}{k} C_B \\ &\int_{\eta_*}^{\eta} d\zeta \left[ \frac{e^{-ik\eta}}{\eta} \frac{e^{ik\zeta}}{\zeta} - \frac{e^{ik\eta}}{\eta} \frac{e^{-ik\zeta}}{\zeta} \right] \times \\ &\begin{cases} A_1 L^{-7/2}(\zeta) & \text{for } 0 < k < L^{-1}(\zeta) \\ A_2 L^{(\alpha-7)/2}(\zeta) k^{\alpha/2} & \text{for } L^{-1}(\zeta) < k < \lambda^{-1}(\zeta) \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (79)$$

For a given mode  $k$ , the time integration has to be divided in different ranges: if the mode satisfies initially  $k < L_*^{-1}$ , then there exists a time  $\eta_L(k)$  at which the correlation length overcomes  $k^{-1}$ , and we have  $k \geq L^{-1}(\eta)$  for  $\eta \geq \eta_L(k)$ . From this time on, one has to consider the small scale part of the source spectrum. Moreover, the modes that are super-horizon at beginning  $k \leq \eta_*^{-1}$  have to be considered separately, since in this case the integrand does not oscillate until horizon crossing. However, since the magnetic correlation length  $L(\eta)$  is increasing with time, the above integrals are in any case dominated by their values at the lower bound  $\eta_*$ . We split the solution in three  $k$  regions, and we finally obtain

$$\begin{aligned} h(k, \eta) &\simeq \frac{4\pi G}{H_0^2 \Omega_{\text{rad}}} \frac{i}{k} C_B \\ &\begin{cases} A_1 L_*^{-7/2} \frac{2}{7\gamma} \left[ \frac{e^{-ik\eta}}{\eta} - \frac{e^{ik\eta}}{\eta} \right] & \text{for } 0 < k < \eta_*^{-1} \\ A_1 L_*^{-7/2} \frac{i}{k\eta_*} \left[ \frac{e^{-ik(\eta-\eta_*)}}{\eta} + \frac{e^{ik(\eta-\eta_*)}}{\eta} \right] & \text{for } \eta_*^{-1} < k < L_*^{-1} \\ A_2 L_*^{-7/2} (k L_*)^{\alpha/2} \frac{i}{k\eta_*} \left[ \frac{e^{-ik(\eta-\eta_*)}}{\eta} + \frac{e^{ik(\eta-\eta_*)}}{\eta} \right] & \text{for } L_*^{-1} < k < \lambda_*^{-1} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (80)$$

This solution has to be matched with the solution of the homogeneous equation at the time the source ceases to be active: that is, for every mode  $k$ , the time  $\eta_\lambda(k)$  at which the dissipation scale has grown bigger than  $k^{-1}$ . However, the above solution does not depend on  $\eta_\lambda(k)$ , being dominated by the lower bound  $\eta_*$ , and has the same

time dependence as a free propagating wave. Therefore, the above solution is simply valid also for  $\eta > \eta_\lambda(k)$ .

We want to evaluate the gravitational wave energy density per logarithmic unit of frequency given in Eq. (48). For this we have to average  $|\dot{h}|^2$  over several periods. Substituting the expression of the scale factor  $a(\eta) = H_0 \sqrt{\Omega_{\text{rad}}} \eta$  we find

$$|\dot{h}|^2(k, \eta) = \frac{(4\pi G)^2}{H_0^2 \Omega_{\text{rad}}} C_B^2 \frac{2}{a^2(\eta)} \begin{cases} A_1^2 L_*^{-7} \left(\frac{2}{7\gamma}\right)^2 & \text{for } 0 < k < \eta_*^{-1} \\ A_1^2 L_*^{-7} \left(\frac{1}{k\eta_*}\right)^2 & \text{for } \eta_*^{-1} < k < L_*^{-1} \\ A_2^2 L_*^{-7} (kL_*)^\alpha \left(\frac{1}{k\eta_*}\right)^2 & \text{for } L_*^{-1} < k < \lambda_*^{-1} \\ 0 & \text{otherwise.} \end{cases} \quad (81)$$

Here we have used that  $\dot{h} \simeq kh$  on sub-horizon scales. The gravitational wave energy density is now given by (48). We substitute the constant  $C_B$  using expression (63), and since  $\Omega_B L^5$  is time independent, we can insert its value at the initial time,  $\Omega_B(\eta_*) L_*^5$ . These manipulations lead to (see fig. 4)

$$\frac{d\Omega_G(k, \eta_0)}{d\log(k)} \simeq 75\pi^2 \left(\frac{\alpha+3}{\alpha-2}\right)^2 \frac{\Omega_B^2(\eta_*)}{\Omega_{\text{rad}}} \begin{cases} A_1^2 \left(\frac{2}{7\gamma}\right)^2 (L_* k)^3 & \text{for } 0 < k < \eta_*^{-1} \\ A_1^2 \left(\frac{L_*}{\eta_*}\right)^2 (L_* k) & \text{for } \eta_*^{-1} < k < L_*^{-1} \\ A_2^2 \left(\frac{L_*}{\eta_*}\right)^2 (L_* k)^{\alpha+1} & \text{for } L_*^{-1} < k < \lambda_*^{-1} \\ 0 & \text{otherwise.} \end{cases} \quad (82)$$

The total energy density can be approximated by

$$\Omega_G = \int_0^\infty \frac{dk}{k} \frac{d\Omega_G(k, \eta_0)}{d\log(k)} \quad (83)$$

$$\simeq 75\pi^2 \left(\frac{\alpha+3}{\alpha-2}\right)^2 \left(A_1^2 - \frac{A_2^2}{\alpha+1}\right) \left(\frac{L_*}{\eta_*}\right)^2 \frac{\Omega_B^2(\eta_*)}{\Omega_{\text{rad}}},$$

where we have neglected a term  $(L_*/\eta_*)^3$ , given that  $L_* \leq \eta_*$ . If the magnetic field at small scales develops an Iroshnikov Kraichnan spectrum corresponding to  $\alpha = -7/2$ , the above quantity becomes

$$\Omega_G \simeq \frac{24}{77}\pi^2 \left(\frac{L_*}{\eta_*}\right)^2 \frac{\Omega_B^2(\eta_*)}{\Omega_{\text{rad}}}. \quad (84)$$

#### D. Limits on the magnetic field intensity

In paper [18], we found strong constraints on the amplitude of a primordial magnetic field by applying the nucleosynthesis bound on the gravitational waves generated by the magnetic field. The limits apply to magnetic fields created both during inflation and during a phase transition, and are formulated in terms of the magnetic field

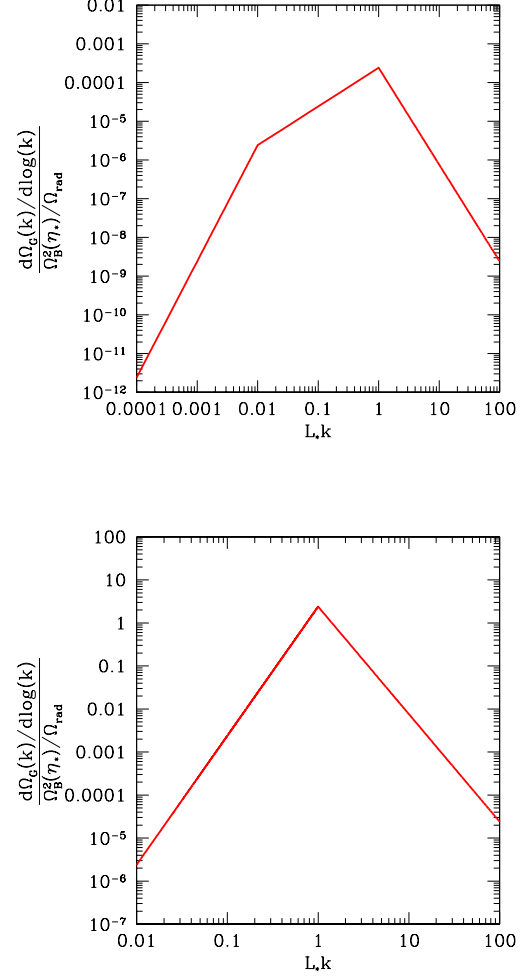


FIG. 4: The gravitational wave energy density today, normalised to  $\Omega_B^2/\Omega_{\text{rad}}$ , as a function of  $x = kL_*$  is shown for  $L_* \mathcal{H}_* = 0.01$  in the top panel and for  $L_* \mathcal{H}_* = 1$  in the bottom panel. On scales  $k < 1/\eta_*$  the spectrum behaves as  $k^3$ , then it flattens into linear growth up to the correlation scale  $L_*^{-1}$ . On smaller scales it decays like  $k^{\alpha+1}$ . In this plot  $\alpha = -7/2$  and  $\gamma = 2/7$  are chosen.

intensity  $B_\Lambda$  present today on the scale  $\Lambda = 0.1$  Mpc. This is the relevant scale for the fields observed in clusters of galaxies. Since the magnetic field generation takes place long before  $\eta = 0.1$  Mpc  $\simeq 10^{13}$  sec, the mode  $\Lambda$  is super-horizon at the moment of creation.

If the magnetic field is generated at inflation, it is characterised by a simple power law spectrum  $k^n$  with  $n > -3$ , up to an upper cutoff which corresponds to the time at which inflation ends,  $k < \eta_*^{-1}$ , where  $\eta_* \simeq 8 \times 10^{-9}$  sec is a typical value. The generation of magnetic fields during a phase transition instead is a causal process, leading mainly to fields on scales smaller than the size of the horizon at the moment the field is created. In the case

of the electroweak phase transition at 100 GeV, this corresponds to  $\eta_* \simeq 10^5$  sec.

In Ref. [18] we have neglected sub-horizon fields, which cannot propagate to larger scales unless they are helical and subject to an inverse cascade. Motivated by inflation, we have imposed a cutoff scale  $\eta_*^{-1}$  in the magnetic power spectrum corresponding to the horizon size at the phase transition. We have argued, that this leads to conservative bounds for the magnetic fields. Here we show that taking into account also the sub-horizon contribution strengthens the limits somewhat.

As we have seen, gravitational wave production takes place also inside the horizon. Moreover, under realistic circumstances, most of the magnetic energy is stored on these scales. Therefore, we now recalculate the bounds obtained in [18] accounting also for sub-horizon modes. The results we find below reduce to those of Ref. [18] in the limiting case  $L_* \simeq \eta_*$ . As for the previous sections, our results reside on the assumption that the large scale part of the magnetic spectrum is persistent.

The nucleosynthesis limit on any additional radiation-like form of energy comes from the fact that this latter may not significantly change the expansion law of the universe during nucleosynthesis. The maximum allowed additional energy density redshifted to today is about  $\Omega_N \lesssim 10^{-5}$  [21]. We impose this bound on the gravitational wave energy density generated by the magnetic field. To be specific, we now fix the value  $\alpha = -7/2$ , but this choice does not affect the final result in a relevant way. From Eq. (84), we have therefore

$$\Omega_B^2(\eta_*) \lesssim \frac{77}{24\pi^2} \times 10^{-5} \Omega_{\text{rad}} \left( \frac{\eta_*}{L_*} \right)^2. \quad (85)$$

The upper bound on the total magnetic energy density is increased for  $L_* \leq \eta_*$ . We want to formulate the bound in terms of the magnetic field amplitude on the scale  $\Lambda = 0.1$  Mpc. At this aim, we perform a volume average of the field in a region of size  $\Lambda^3$ . We convolve the field with a Gaussian window function

$$\mathbf{B}_\Lambda(\mathbf{x}) = \int \frac{d^3y}{V_\Lambda} \mathbf{B}(\mathbf{x} + \mathbf{y}) \exp\left(-\frac{y^2}{2\Lambda^2}\right), \quad (86)$$

and we define the magnetic field energy density smoothed on a scale  $\Lambda$ ,  $B_\Lambda^2 = \langle \mathbf{B}_\Lambda(\mathbf{x}) \cdot \mathbf{B}_\Lambda(\mathbf{x}) \rangle$ . We have therefore

$$B_\Lambda^2 = \frac{1}{\pi^2} \int_0^\infty dk k^2 P_B(k) \exp\left(-\frac{\Lambda^2 k^2}{2}\right). \quad (87)$$

Since  $\Lambda \gg L_*$ , the above integral is dominated by the large scale part of the spectrum  $kL_* \leq 1$ , and inserting Eq. (63) we obtain finally

$$B_\Lambda^2 \simeq \frac{120}{11} \sqrt{2\pi} \rho_c \Omega_B(\eta_*) \left( \frac{L_*}{\Lambda} \right)^5, \quad (88)$$

where again we have taken  $\Omega_B L^5$  at the initial time. We can now use Eq. (85) to derive an upper bound on the

magnetic field intensity  $B_\Lambda$ :

$$\frac{B_\Lambda}{10^{-6} \text{ Gauss}} \lesssim \left( \frac{L_*}{\eta_*} \right)^2 \left( \frac{\eta_*}{\Lambda} \right)^{5/2}. \quad (89)$$

Comparing with Eq. (33) of Ref. [18], we see that the bound is strengthened by a factor  $\left( \frac{L_*}{\eta_*} \right)^2$ . The reason for that is twofold. First, the limit on  $\Omega_B$  is increased by a factor  $\eta_*/L_*$  when reducing  $L_*$  for fixed  $\eta_*$ . On the other hand, the magnetic field is peaked at the smaller scale  $L_*$ : this appears in the value of  $B_\Lambda^2$  through the factor  $(L_*/\Lambda)^5 = (L_*/\eta_*)^5 (\eta_*/\Lambda)^5$ .

In the case of the electroweak phase transition at 100 GeV, for which  $\eta_* \simeq 10^5$  sec, we find

$$B_\Lambda \lesssim 10^{-26} \left( \frac{L_*}{\eta_*} \right)^2 \text{ Gauss}. \quad (90)$$

If the phase transition is first order and proceeds via bubble nucleation, we expect the correlation length to be given by the size of the largest bubbles which has been estimated as  $L_* \sim 0.01\eta_*$ , leading to the gravity wave spectrum shown in the top panel of Fig. 4. However, if the phase transition is second order, the magnetic field inside one horizon volume is created smoothly and has enough time to align itself. In this case we therefore expect a correlation length of the order of  $L_* \simeq \eta_*$ , leading to the gravity wave spectrum shown in the bottom panel of Fig. 4.

## V. COMPARISON OF THE TWO CASES

We have calculated gravity wave production from two sources which are both represented by divergence free vector fields. One of them (the magnetic field) is persistent over many Hubble times while turbulence decays within a fraction of one Hubble time. We would naturally expect the former to produce more gravity waves than the latter. However, this is not properly the case. Here we compare the efficiency of the two sources, turbulent and magnetic, in detail. Let us first compare the total gravitational wave energy produced, as a function of the total energy available in the sources. For this, we consider Eqs. (57) and (84), where the first one applies for the largest eddy velocity  $1/2 \leq v_L \leq 1/\sqrt{3}$  and the second one for the value of the magnetic field spectrum  $\alpha = -7/2$ . Denoting the quantities concerning turbulence by a superscript  $T$  and those concerning magnetic fields by  $B$ , we have

$$\frac{\Omega_G^T}{\Omega_G^B} \simeq 0.02 \left( \frac{L_*^T}{L_*^B} \right)^2 \frac{\left( \frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} \right)}{\left( \frac{\Omega_B(\eta_*)}{\Omega_{\text{rad}}} \right)^2}. \quad (91)$$

The gravitational wave energy density generated by magnetic fields depends quadratically on the total energy of the magnetic source, while for turbulence the relation is linear. This is due to the presence of the factor

$v_L^{-2} = \frac{2}{3} \frac{\Omega_{\text{rad}}(\eta_*)}{\Omega_T(\eta_*)}$  in the amplitude of the gravitational wave turbulent spectrum (54). If the turbulent and magnetic correlation scales as well as the total energy available in the sources are comparable, the efficiency in generating gravitational waves is similar. For  $v_L = 1/\sqrt{3}$ , we find from Eq. (91)  $\Omega_G^T/\Omega_G^B \simeq 0.1$ , roughly the same order of magnitude within our accuracy. We find no significant domination of the magnetic field induced total gravity wave energy.

We now compare the gravitational wave energy density spectra, in order to analyse the efficiency of the sources as a function of wavenumber  $k$ . We want to relate Eqs. (54) and (82). For this we set the correlation lengths to the same value  $L_*^T = L_*^B = L_*$ . Therefore we take  $L_*\mathcal{H}_* \simeq 0.01$ , as in the case of turbulence generated after a phase transition (*cf.* section III). Moreover, we further impose that the initial available energy densities are comparable,  $\Omega_T(\eta_*)/\Omega_{\text{rad}}(\eta_*) \simeq \Omega_B(\eta_*)/\Omega_{\text{rad}}$ . Applying the reasonable values  $v_L = 1/\sqrt{3}$ ,  $\alpha = -7/2$ ,  $\gamma = 2/7$ , the ratio of Eqs. (54) and (82) becomes ( $x = kL_*$ )

$$\frac{\left(\frac{d\Omega_G^T(k, \eta_0)}{d\log(k)}\right)}{\left(\frac{d\Omega_G^B(k, \eta_0)}{d\log(k)}\right)} \simeq 0.1 \begin{cases} (L_*\mathcal{H}_*)^2 & \text{for } 0 < x < L_*\mathcal{H}_* \\ x^2 & \text{for } L_*\mathcal{H}_* < x < 1 \\ \frac{10}{7} x^{1/2} & \text{for } 1 < x < (2v_L)^3 \\ \frac{40}{21} x^{-1/6} & \text{for } (2v_L)^3 < x < L_*/\lambda_* \\ 0 & \text{otherwise.} \end{cases} \quad (92)$$

From the above expression we can conclude that, on super-horizon wavelengths  $x < L_*\mathcal{H}_*$ ,  $k < \eta_*^{-1}$ , the energy density of gravitational waves from turbulence

is smaller than the one from magnetic fields by a factor  $0.1(L_*\mathcal{H}_*)^2 \simeq 10^{-5}$ . For sub-horizon modes, the ratio is increasing and reaches the maximum value at  $k_{\text{max}} = (2v_L)^3/L_*$ , where the two amplitudes are comparable  $\frac{d\Omega_G^T(k, \eta_0)}{d\log(k)} / \frac{d\Omega_G^B(k, \eta_0)}{d\log(k)} \simeq 0.1$ . Since the spectrum peaks at  $k = 1/L_* \simeq k_{\text{max}}$ , also the total energy densities are comparable.

We hence find that on super-horizon scales magnetic fields are more efficient than turbulence in generating gravitational waves. This is due to the difference of the time intervals over which the sources act, and to the characteristic way in which a radiation-like source converts energy into gravitational waves. In the integrals of Eq. (79), the Green's function oscillates with frequency  $k$  and decays as a power law. Therefore, super-horizon modes,  $k\eta \leq 1$  are more efficiently converted into gravitational waves than sub-horizon modes. Since the magnetic field is active for many Hubble times (approximately from its generation up the epoch of matter-radiation equality), the magnetic super-horizon modes generate gravity waves until they cross the horizon (subsequent generation can be neglected). Turbulence, on the other hand, is active as a source of gravitational waves only for a time interval much shorter than one Hubble time  $\tau_L \ll \mathcal{H}_*^{-1}$ . Most turbulent modes that start super-horizon do not have the time to convert their energy in gravitational waves. This leads to the suppression factor  $(L_*/\eta_*)^2 = (L_*\mathcal{H}_*)^2$  for the gravity wave spectrum from turbulence.

The expression for the turbulence to magnetic field ratio in full generality (with  $L_* = L^T = L_*^B$ ) is

$$\frac{\frac{d\Omega_G^T(k, \eta_0)}{d\log(k)}}{\frac{d\Omega_G^B(k, \eta_0)}{d\log(k)}} \simeq \frac{3}{800\pi^3} \left(\frac{\alpha-2}{\alpha+3}\right)^2 \frac{1}{v_L^2} \frac{\left(\frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)}\right)^2}{\left(\frac{\Omega_B(\eta_*)}{\Omega_{\text{rad}}}\right)^2} \begin{cases} \frac{1}{A_1^2} \left(\frac{7\gamma}{2}\right)^2 (L_*\mathcal{H}_*)^2 & \text{for } 0 < x < L_*\mathcal{H}_* \\ \frac{1}{A_1^2} x^2 & \text{for } L_*\mathcal{H}_* < x < 1 \\ \frac{1}{A_2^2} x^{-(\alpha+3)} & \text{for } 1 < x < (2v_L)^3 \\ \frac{4v_L^2}{A_2^2} x^{-(\alpha+11/3)} & \text{for } (2v_L)^3 < x < L_*/\lambda_* \\ 0 & \text{otherwise.} \end{cases} \quad (93)$$

## VI. CONCLUSIONS

We have analysed the generation of a gravitational wave background by a period of turbulence in the primordial universe, arising after a strongly first order phase transition, and by the presence of a primordial magnetic field. Since the early universe is an ionised plasma, treating pure Kolmogorov turbulence on its own is probably not realistic. However, given the previous literature on this subject, the analysis in Section III is essential [9, 10, 11, 12]. We have present the more realistic case of

MHD turbulence in Section IV, and a comparison of the two cases in Section V.

The turbulent velocity field and the (causally created) magnetic field are necessarily uncorrelated on scales larger than a given correlation scale, which is at best equal to the cosmological horizon. We have demonstrated that this fact, together with the property of vanishing divergence shared by the two fields, implies that their power spectra are blue on scales larger than the correlation scale. This was already known in the case of causal magnetic fields [32], but is new for turbulence. A phase of

turbulence in the primordial universe is inevitably characterised by a Batchelor energy spectrum. A Saffman spectrum is excluded, since it relies on long range correlations that can be generated by the pressure fields [17]. On small scales, we have assumed Kolmogorov two thirds law for the turbulent velocity correlation, and we have derived the turbulent spectrum from direct integration of the correlation function. We have used the asymptotic behavior of the turbulence power spectrum right up to the peak region. Therefore, the amplitude of our result is probably not very reliable. We plan to improve this with a numerical analysis [33].

Since turbulence is an incoherent source, the analysis of the gravitational waves generated by it is not straightforward. In Section III D we propose a procedure where the time dependence of the turbulent flow is not lost in the ensemble averaging. This time dependence is important for the correct determination of the induced gravitational waves. Our method to analyse the formation of gravitational waves is based on an heuristic model of the oscillations of the random velocity field. This model is inspired by Richardson's energy cascade through eddies of different sizes.

Our work corrects an error in the dispersion relation found in previous papers [9, 10, 11, 12] and provides the correct dispersion relation for the free propagation of the gravitational waves, after the turbulent source ceases to be active. In Ref. [10] the sub-horizon power spectrum of the gravitational waves, at the time when the turbulent source ceases to be active, is given in Eq. (44) as a function of  $k$ . The component  $\langle h_{ij}(\mathbf{k}, \eta_*) h_{ij}^*(\mathbf{q}, \eta_*) \rangle \propto \delta^3(\mathbf{k} - \mathbf{q})$  of the spectrum has momentum  $\mathbf{k}$ . However, then they set the gravitational wave frequency at time  $\eta_*$  to  $\omega_k$ , the oscillation frequency of the turbulent source,

$$\omega_k = (v_L/L^{1/3})k^{2/3}. \quad (94)$$

This means that the gravitational wave dispersion relation at time  $\eta_*$  is the same as the dispersion relation of the turbulent source. However, for  $\eta > \eta_*$  the gravitational wave propagates freely, and the dispersion relation should simply be  $\omega = k$ . For  $\eta > \eta_*$  the gravitational wave power spectrum changes only by redshifting with the expansion of the universe. Therefore, in this procedure it is not clear how one can recover the correct dispersion relation for gravity waves at  $\eta > \eta_*$ .

The origin of this mistake is the idea that the gravity wave frequency is determined by the frequency and not by the wave number of the source. This is certainly true for spatially small sources with infinite lifetime [34]. However, in Appendix A we show that for a gravitational wave source which is localised in time but not in space, the frequency of the emitted gravitational wave is determined by the wavenumber of the source (the generic relation between the frequency of the gravity wave and the frequency and the wave number of its source are derived in Ref. [35]). This is particularly important for turbulence, since the turbulent eddies do not spin at the speed of light. As a consequence, we find that the energy

power spectrum of the induced gravitational waves peaks at a frequency corresponding to the *size* of the largest turbulent eddies (the correlation scale), and not to their frequency. The spectrum continues to both larger and smaller wavelengths with different slopes.

Usually, the nucleating bubbles of true vacuum also generate anisotropic stresses in the fluid and therefore lead to the production of gravity waves. The spectrum of these gravity waves from the bubbles also peaks at frequency  $L_*^{-1}$ . Its amplitude has been estimated for example in Ref. [9].

In the case of magnetic fields, we obtain a blue power spectrum on large scales, leaving as free parameters both the slope of the power spectrum on small scales, and the growth of the correlation length with time. Assuming the existence of the invariant quantity  $\langle B^2 \rangle L^5$  analogous to Loitsyansky's integral, we obtain that the comoving anisotropic stress depends on time only via the magnetic correlation length. We have solved the evolution equation for gravitational waves sourced by magnetic anisotropic stresses accounting for the evolution of the magnetic correlation length. This has allowed us to derive new limits for the amplitude of the magnetic field as a function of the correlation scale, extending the results of Ref. [18] to correlation scales which are smaller than the horizon.

We conclude our analysis with a comparison of the efficiency of turbulent and magnetic sources. When imposing that the total energies available in the sources and their correlation lengths be comparable, we find that the total gravitational wave energy densities are of the same order, but the intensity of gravitational waves generated by magnetic fields on super-horizon scales is enhanced with respect to that of gravitational waves from turbulence by a factor  $(L_* \mathcal{H}_*)^2$ . We can conclude that a coherent, relativistic, long acting source as the magnetic field is more efficient in generating gravitational waves on super-horizon scales, than a source which acts only over a brief period of time like turbulence.

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### APPENDIX A

In this appendix we want to clarify that, for a stochastic gravitational wave source which is homogeneously distributed in space and which has a time duration comparable to its characteristic oscillation period, the frequency of the generated gravity wave is connected to the wavevector  $\mathbf{k}$  (hence to the typical size) of the source and not to its frequency. The calculation is already performed for the particular case of the turbulence in section III D,



but it is repeated here in more generality for clarification. We neglect cosmological expansion since it is of no relevance for our argument and consider simply a Minkowski background. The wave equation

$$(\partial_t^2 - \Delta)h_{ij}(\mathbf{x}, t) = 8\pi G\Pi_{ij}(\mathbf{x}, t), \quad (\text{A1})$$

can be solved by the retarded potential

$$h_{ij}(\mathbf{x}, t) = 8\pi G \int d^3x' \frac{\Pi_{ij}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|}. \quad (\text{A2})$$

For a long lived source confined in some small spatial region, one may consider the usual wave-zone approximation. In the wave zone, that is at distances much larger than the dimension of the source, the field emitted by the source looks like an outgoing spherical wave with wavenumber  $k = \omega$ , where  $\omega$  is the source frequency  $\Pi_{ij}(\mathbf{x}, t) = \Pi_{ij}(\mathbf{x})e^{i\omega t}$  [34].

However, we are treating the case of a stochastic, statistically homogeneous cosmological source. The source is not stationary, but characterized by a well defined time interval of action. Therefore, in order to solve the above equation we first Fourier transform it in space

$$(\partial_t^2 + k^2)h_{ij}(\mathbf{k}, t) = 8\pi G\Pi_{ij}(\mathbf{k}, t). \quad (\text{A3})$$

The retarded Green's function of the above operator (the Fourier transform of the Green function used in eq. (A2)) is simply

$$G_R(k, t) = \Theta(t) \frac{\sin(kt)}{k}, \quad (\text{A4})$$

where  $\Theta(t)$  is the Heaviside function. If the source starts emitting at a time  $t_{\text{in}}$  we have

$$h_{ij}(\mathbf{k}, t) = 8\pi G \int_{t_{\text{in}}}^t dt' \frac{\sin(k(t - t'))}{k} \Pi_{ij}(\mathbf{k}, t'). \quad (\text{A5})$$

At the time  $t_{\text{fin}}$ , the gravitational wave is given by

$$\begin{aligned} h_{ij}(\mathbf{k}, t_{\text{fin}}) &= 8\pi G \int_{t_{\text{in}}}^{t_{\text{fin}}} dt' \frac{\sin(k(t_{\text{fin}} - t'))}{k} \Pi_{ij}(\mathbf{k}, t') \\ \dot{h}_{ij}(\mathbf{k}, t_{\text{fin}}) &= 8\pi G \int_{t_{\text{in}}}^{t_{\text{fin}}} dt' \cos(k(t_{\text{fin}} - t')) \Pi_{ij}(\mathbf{k}, t'). \end{aligned}$$

After the source stops emitting, the gravitational wave background obeys the free wave equation  $(\partial_t^2 + k^2)h_{ij}(\mathbf{k}, t) = 0$ , with solution

$$h_{ij}(\mathbf{k}, t > t_{\text{fin}}) = A_{ij} \frac{\sin(k(t - t_{\text{fin}}))}{k} + B_{ij} \cos(k(t - t_{\text{fin}})). \quad (\text{A6})$$

Matching the free wave and the sourced wave at time  $t_{\text{fin}}$  we obtain

$$\begin{aligned} A_{ij}(\mathbf{k}) &= 8\pi G \int_{t_{\text{in}}}^{t_{\text{fin}}} dt' \cos(k(t_{\text{fin}} - t')) \Pi_{ij}(\mathbf{k}, t') \\ B_{ij}(\mathbf{k}) &= 8\pi G \int_{t_{\text{in}}}^{t_{\text{fin}}} dt' \frac{\sin(k(t_{\text{fin}} - t'))}{k} \Pi_{ij}(\mathbf{k}, t') \end{aligned} \quad (\text{A7})$$

Regardless of the particular time dependence of the source  $\Pi_{ij}(\mathbf{k}, t')$ , the free gravitational field (A6) oscillates with the frequency  $\omega = k$  corresponding to the wavenumber of the source. If the source is itself oscillating, but it obeys a non-trivial dispersion relation  $\omega(k) \neq k$ , then the source frequency intervenes in the amplitude of the free propagating wave. Only if the duration  $t_{\text{fin}} - t_{\text{in}}$  is much larger than the period  $1/\omega$ , the above integrals can be approximated by  $\delta(k - \omega)$  and the wave inherits the frequency of the source [35].

The same result can also be obtained from Eq. (A2), using that  $|\mathbf{x} - \mathbf{x}'|$  has to lie in the interval  $[t - t_{\text{fin}}, t - t_{\text{in}}]$ .

This has been misrepresented in previous literature (see e.g. [9, 11]). It is not very important for the results obtained as long as  $v_L = \mathcal{O}(1)$ , but it is an interesting conceptual point: A stochastic spatially homogeneous source for gravity waves which has a lifetime comparable to its characteristic oscillation period (as it is the case for turbulence) imprints its *wave number* and not its frequency in the gravity waves it produces. On the other hand, a stationary and isolated source always imprints its *frequency* and not its wave number.

## APPENDIX B

In this appendix we rewrite the spectra of the gravitational waves generated by turbulence in terms of the phase transition parameters as they are conventionally defined. We conform to the notations of Refs. [11, 12] which are commonly used in the literature. We denote  $u_S = v_L$  the velocity of the largest eddies, so that the total kinetic energy of the turbulence is  $\frac{\Omega_T(\eta_*)}{\Omega_{\text{rad}}(\eta_*)} \simeq \frac{2}{3}u_S^2$ . The duration of the phase transition is  $\beta^{-1} = a_*\Delta\eta_*$ , and  $H_*$  is the physical Hubble parameter at the phase transition. The frequency of the gravitational waves is denoted  $f = k$ , and  $f_S = \omega_L/a_*$  is the physical turnover frequency of the largest eddies scaled to today. The peak frequency given in Eq. (55) corresponds to  $k_L \simeq f_S/u_S$ . We call  $f_{\text{max}} = 1/\lambda$  the maximal frequency corresponding to the dissipation scale. Finally, the gravitational wave energy density per logarithmic unit of frequency is denoted  $\Omega_{GW}(f)$ . We now express our results in terms of  $f_S$ ,  $u_S$ ,  $H_*$ ,  $\beta$  and  $v_b$ .

Instead of considering the full expression (52), we find it more useful to rewrite the approximated formulas (53) and (54). In the case  $u_S \leq 1/2$ , Eq. (53), we find

$$\Omega_{GW}(f) \simeq \frac{\Omega_{\text{rad}}}{2\pi} \left( \frac{H_*}{\beta} \right)^2 v_b^2 \begin{cases} \frac{u_S^5}{4} \left( \frac{f}{f_S} \right)^3 & \text{for } 0 < f < 2f_S \\ u_S^5 \frac{f}{f_S} & \text{for } 2f_S < f < f_S/u_S \\ u_S^{4/3} \left( \frac{f_S}{f} \right)^{8/3} & \text{for } f_S/u_S < f < f_{\text{max}} \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B1})$$

In the case  $u_S \geq 1/2$ , Eq. (54), we find

$$\Omega_{GW}(f) \simeq \frac{\Omega_{\text{rad}}}{2\pi} \left( \frac{H_*}{\beta} \right)^2 v_b^2 \begin{cases} \frac{u_S^5}{4} \left( \frac{f}{f_S} \right)^3 & \text{for } 0 < f < f_S/u_S \\ \frac{1}{4} \left( \frac{f_S}{f} \right)^2 & \text{for } f_S/u_S < f < 8u_S^2 f_S \\ u_S^{4/3} \left( \frac{f_S}{f} \right)^{8/3} & \text{for } 8u_S^2 f_S < f < f_{\text{max}} \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B2})$$

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